Problems and Solutions

in Mathematics, Physics and Applied Sciences

compiled by C. Bond Vol.1 no.1

A Trigonometric Series

It is known that a trigonometric series can be used to approximate functions which cannot be approximated by polynomials. In particular, functions which are constant over an interval or which have discontinuities can, under suitable restrictions, be represented by trigonometric series but not by polynomials.

The branch of mathematics known as Fourier analysis has been developed to solve problems in representing and analyzing such functions, and we are interested in 'peeking under' the standard assumptions of Fourier analysis to discover some of its algebraic foundations. Hence the following will not involve integrations or any of the properties of orthogonal functions, and will instead proceed along the lines originally laid out by Mr. Fourier.

The question to be answered is whether we can solve

$$1 = a\cos x + b\cos 3x + c\cos 5x + d\cos 7x\cdots \tag{1}$$

over some interval about $x = 0$. That is, can we determine the coefficients a, b, c, etc. so that Equation 1 is satisfied?[1]

Our strategy, following Fourier, is to solve a sequence of finite systems of equations involving increasing numbers of unknowns. From each of these systems the coefficients can be evaluated and recursion rules developed to compute values for the infinite system implied by the above.

[1] Note that the equation is an even function (no sine terms) with quarter wave symmetry. We expect our solution to be periodic and to be valid only within a restricted portion of the full period.

To start, we will determine the value of a. The first order system is simply

$$1 = a \cos x$$

At $x = 0$, this reduces to $a = 1$.

To keep track of the values of the variables a, b, c, etc. for different sized systems of equations, we will adopt the convention that a_1 represents the value of a found in solving the first order system. Similarly, a_j represents the value of a found in solving a j^{th} order system. A corresponding interpretation for the symbols b_4, c_k, etc. should be made. Thus, $a_1 = 1$. In subsequent equations we will sometimes replace 1 with 1^2 for reasons of symmetry.

Next form a second order system by introducing the second variable, b, to form one equation and differentiate this equation twice to form a second equation in the two unknowns.[2]

$$\begin{aligned} 1 &= a_2 \cos x + b_2 \cos 3x \\ 0 &= -a_2 \cos x - 3^2 b_2 \cos 3x \end{aligned}$$

At $x = 0$ this reduces to

$$\begin{aligned} 1 &= a_2 + b_2 \\ 0 &= a_2 + 3^2 b_2 \end{aligned}$$

which gives

$$a_2 = \frac{3^2}{3^2 - 1^2}$$

At this point, it is instructive to examine the solution process for a system with, say, 5 unknowns.

$$\begin{aligned} 1 &= a_5 + b_5 + c_5 + d_5 + e_5 \\ 0 &= a_5 + 3^2 b_5 + 5^2 c_5 + 7^2 d_5 + 9^2 e_5 \\ 0 &= a_5 + 3^4 b_5 + 5^4 c_5 + 7^4 d_5 + 9^4 e_5 \\ 0 &= a_5 + 3^6 b_5 + 5^6 c_5 + 7^6 d_5 + 9^6 e_5 \\ 0 &= a_5 + 3^8 b_5 + 5^8 c_5 + 7^8 d_5 + 9^8 e_5 \end{aligned}$$

[2]Note that a single differentiation produces an equation in sines which is identically zero at $x = 0$ and cannot be used to form the second order system.

From this, we eliminate e_5 to form a system of four equations in four unknowns. The method used is to multiply the first equation by 9^2 and subtract the second to form a new equation with e_5 eliminated. Then multiply the second equation by 9^2 and subtract the third forming an additional equation. Continue until the following result is obtained.

$$\begin{aligned}
9^2 &= (9^2-1^2)a_5 + (9^2-3^2)b_5 + (9^2-5^2)c_5 + (9^2-7^2)d_5 \qquad (2)\\
0 &= (9^2-1^2)a_5 + 3^2(9^2-3^2)b_5 + 5^2(9^2-5^2)c_5 + 7^2(9^2-7^2)d_5\\
0 &= (9^2-1^2)a_5 + 3^4(9^2-3^2)b_5 + 5^4(9^2-5^2)c_5 + 7^4(9^2-7^2)d_5\\
0 &= (9^2-1^2)a_5 + 3^6(9^2-3^2)b_5 + 5^6(9^2-5^2)c_5 + 7^6(9^2-7^2)d_5
\end{aligned}$$

Similarly, we can eliminate d and reduce this to a system of three equations in three unknowns by multiplying the first equation by 7^2 and subtracting the second to form the first equation in the new system, etc. This results in

$$\begin{aligned}
7^2 9^2 &= (9^2-1^2)(7^2-1^2)a_5 + (9^2-3^2)(7^2-3^2)b_5 + (9^2-5^2)(7^2-5^2)c_5\\
0 &= (9^2-1^2)(7^2-1^2)a_5 + 3^2(9^2-3^2)(7^2-3^2)b_5 + 5^2(9^2-5^2)(7^2-5^2)c_5\\
0 &= (9^2-1^2)(7^2-1^2)a_5 + 3^4(9^2-3^2)(7^2-3^2)b_5 + 5^4(9^2-5^2)(7^2-5^2)c_5
\end{aligned}$$

and, on continued reduction we find

$$a_5 = \frac{3^2}{3^2-1^2}\cdot\frac{5^2}{5^2-1^2}\cdot\frac{7^2}{7^2-1^2}\cdot\frac{9^2}{9^2-1^2}$$

By induction, the value of a_6 (for a sixth order system) would be $11^2/(11^2-1^2)$ times a_5. For an infinite system, a_∞ is

$$a_\infty = a = \frac{3^2}{3^2-1^2}\cdot\frac{5^2}{5^2-1^2}\cdot\frac{7^2}{7^2-1^2}\cdot\frac{9^2}{9^2-1^2}\cdot\frac{11^2}{11^2-1^2}\cdots$$

which can be expressed as a simple product series

$$a = \frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{9\cdot 9}{8\cdot 10}\cdots \qquad (3)$$

Equation 3 is a variation of Wallis' product formula. It's value is known to be

$$a = \frac{4}{\pi}$$

which is the desired value for a in Equation 1.

We will now outline a procedure to evaluate the rest of the variables.

There will be an initial value for each variable that occurs when the variable is first introduced as the size of the system increases. Thus the first appearance of a is identified as a_1. The first value of b is b_2 and of e is e_5. We can determine these initial values from the same systems of equations used previously.

Once we have solved for the initial values of each of the variables, we need only develop a recursive formula for generating the final values from the initial ones.

For example, the initial value of b is found by solving the second order system

$$\begin{aligned} 1 &= a_2 + b_2 \\ 0 &= a_2 + 3^2 b_2 \end{aligned}$$

from which

$$b_2 = -\frac{1^2}{3^2 - 1^2}$$

The initial value of c is found from

$$\begin{aligned} 1 &= a_3 + b_3 + c_3 \\ 0 &= a_3 + 3^2 b_3 + 5^2 c_3 \\ 0 &= a_3 + 3^4 b_3 + 5^4 c_3 \end{aligned}$$

to be

$$c_3 = \frac{1^2}{5^2 - 1^2} \cdot \frac{3^3}{5^2 - 3^2}$$

Continuing in this manner,

$$\begin{aligned} d_4 &= -\frac{1^2}{7^2 - 1^2} \cdot \frac{3^2}{7^2 - 3^2} \cdot \frac{7^2}{7^2 - 5^2} \\ e_5 &= \frac{1^2}{9^2 - 1^2} \cdot \frac{3^2}{9^2 - 3^2} \cdot \frac{5^2}{9^2 - 5^2} \cdot \frac{7^2}{9^2 - 7^2} \end{aligned}$$

and so on.

Observe that the initial value for $d = d_4$ satisfies all equations for the fourth order system. In particular, it satisfies the first equation in the system

$$1 = a_4 + b_4 + c_4 + d_4$$

while d_5, on the other hand, satisfies the first equation in the fifth order system

$$1 = a_5 + b_5 + c_5 + d_5 + e_5$$

Comparing the first equation in the reduced fifth order system (Equation 2) to the first equation in the fourth order system, we see that d_4 must be multiplied by $9^2/(9^2 - 7^2)$ in order to satisfy the fifth order system. Hence,

$$d_5 = d_4 \frac{9^2}{9^2 - 7^2}$$

Similarly,

$$d_6 = d_5 \frac{11^2}{11^2 - 7^2}$$

which follows from the method used to solve the various systems of equations.

We can tabulate the values for all variables obtained similarly as follows:

$$\begin{aligned}
a &= a_1 \cdot \frac{3^2}{3^2 - 1^2} \cdot \frac{5^2}{5^2 - 1^2} \cdot \frac{7^2}{7^2 - 1^2} \cdots \\
b &= -b_2 \cdot \frac{5^2}{5^2 - 3^2} \cdot \frac{7^2}{7^2 - 3^2} \cdot \frac{9^2}{9^2 - 3^2} \cdots \\
c &= c_3 \cdot \frac{7^2}{7^2 - 5^2} \cdot \frac{9^2}{9^2 - 5^2} \cdot \frac{11^2}{11^2 - 5^2} \cdots \\
d &= -d_4 \cdot \frac{9^2}{9^2 - 7^2} \cdot \frac{11^2}{11^2 - 7^2} \cdot \frac{13^2}{13^2 - 7^2} \cdots
\end{aligned}$$

Rewriting these equations as simple product series we have

$$a = \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \frac{7 \cdot 7}{6 \cdot 8} \cdot \frac{9 \cdot 9}{8 \cdot 10} \cdot \frac{11 \cdot 11}{10 \cdot 12} \cdots \qquad (4)$$

$$
\begin{aligned}
b &= -\frac{1\cdot 1}{2\cdot 4}\cdot\frac{5\cdot 5}{2\cdot 8}\cdot\frac{7\cdot 7}{4\cdot 10}\cdot\frac{9\cdot 9}{6\cdot 12}\cdot\frac{11\cdot 11}{8\cdot 14}\cdots \\
c &= \frac{1\cdot 1}{4\cdot 6}\cdot\frac{3\cdot 3}{2\cdot 8}\cdot\frac{7\cdot 7}{2\cdot 12}\cdot\frac{9\cdot 9}{4\cdot 14}\cdot\frac{11\cdot 11}{6\cdot 16}\cdots \\
d &= -\frac{1\cdot 1}{6\cdot 8}\cdot\frac{3\cdot 3}{4\cdot 10}\cdot\frac{5\cdot 5}{2\cdot 12}\cdot\frac{9\cdot 9}{2\cdot 16}\cdot\frac{11\cdot 11}{4\cdot 18}\cdots
\end{aligned}
$$

These can be evaluated by performing modest rearrangements of the terms in each series and reconciling the resulting forms with Wallis' formula. To show how this is done, we will use b as an example.

In the following sequence of equations certain terms of each equation will be exchanged to produce the equation which follows. Note that the exchange of two terms in the numerator or denominator of in infinite product series does not change the value of the series.[3]

The terms to be exchanged in each equation are printed in boldface type so that the effect of the exchange on the following equation can be easily traced.

$$
\begin{aligned}
b &= -\frac{1\cdot 1}{2\cdot 4}\cdot\frac{5\cdot 5}{2\cdot \mathbf{8}}\cdot\frac{7\cdot 7}{\mathbf{4\cdot 10}}\cdot\frac{9\cdot 9}{\mathbf{6\cdot 12}}\cdot\frac{11\cdot 11}{\mathbf{8\cdot 14}}\cdots \\
b &= -\frac{1\cdot 1}{2\cdot 4}\cdot\frac{5\cdot 5}{\mathbf{2\cdot 4}}\cdot\frac{7\cdot 7}{\mathbf{8\cdot 6}}\cdot\frac{9\cdot 9}{\mathbf{10\cdot 8}}\cdot\frac{11\cdot 11}{\mathbf{12\cdot 10}}\cdots \\
b &= -\frac{1\cdot 1}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 2}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot\frac{11\cdot 11}{10\cdot 12}\cdots \qquad (5)
\end{aligned}
$$

Now we multiply b by $3/3$. We are at liberty to multiply any numerator term by 3 and any denominator term by 3. Our choice is dictated by the objective of matching Equation 4 which suggests the second numerator term and the fourth denominator term.

$$b = -\frac{1\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot\frac{11\cdot 11}{10\cdot 12}\cdots$$

Finally, it is clear that

$$3b = -a = -\frac{4}{\pi}$$

from Equation 5 and 4.

[3]Not all operations on infinite products are valid. For example, splitting a term into two parts so as to shift the numerator with respect to the denominator may change the series value.

Similar rearrangements of the equations for the other variables leads to

$$\begin{aligned} a &= \frac{4}{\pi} \\ b &= -\frac{4}{3\pi} \\ c &= \frac{4}{5\pi} \\ d &= -\frac{4}{7\pi} \\ e &= \frac{4}{9\pi} \end{aligned}$$

and so on.

Returning to the original equation, we can now write the solution

$$\frac{\pi}{4} = \cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x \cdots \tag{6}$$

from which

$$1 = \frac{4}{\pi}\cos x - \frac{4}{3\pi}\cos 3x + \frac{4}{5\pi}\cos 5x \cdots \tag{7}$$

Armed with the proof that a constant function can be approximated over an interval by a trigonometric series, we can now assume a form for the solution of a new problem and employ familiar devices to find direct solutions. This is the point at which most textbooks begin the study of Fourier series.