Engineering Notebook

miscellaneous problems and solutions

compiled by C. Bond Vol.1 no.12

A Zone Mapping Problem

The problem to be solved concerns the layout of zones on a digital recording disk. These zones are concentric bands of recorded data within which all cylinders have the same recording rate (frequency). Different zones will then contain material recorded at different rates. Given the option of creating a finite number of zones, it is desired to optimize the zone map so that the maximum capacity of the disk is achieved.

The constraints are that the inner and outer radii over which recording is permitted are fixed and that the track (cylinder) spacing is constant. The maximum packing density is also fixed and will be applied to the inner track in each zone. Hence, the inner radius, R_i, the outer radius, R_o, the number of tracks per inch, TPI, the maximum number of bits per inch, BPI, and the number of zones, N, are given. We are to find the radii which define the zones for maximum capacity.

We will adopt a naming convention based on the drawing in Figure 1, where R_i is

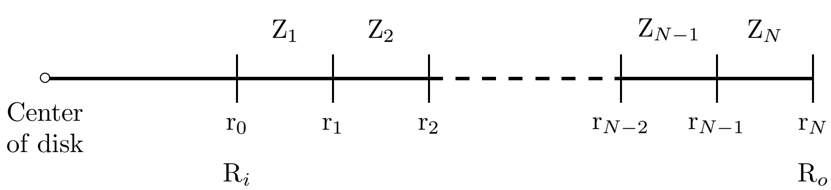

Figure 1: Zone Layout Schematic

a synonym for r_0 and R_o is a synonym for r_N. We will use R_i and R_o when it is important to emphasize that these are given constants and not variables. The zones are shown equally spaced for convenience, but we will prove that this is, in fact, the optimal zone geometry.

To begin, we must write an equation for the capacity of the drive in terms of the named parameters. This is best accomplished by writing an equation for the capacity of each zone and summing all the zones. For Z_1 the capacity is

$$R_i \times BPI \times (r_1 - R_i) \times TPI = C_1$$

where the product of R_i and BPI gives the number of bits per track, and the product of $(r_1 - R_i)$ and TPI gives the number of tracks. It is convenient to define a metric linearly related to the capacity by dividing both sides of the equation by the product of BPI and TPI. Our revised equation is then

$$R_i(r_1 - R_i) = K_1$$

The metric for the problem, related to the capacity of the drive, is

$$R_i(r_1 - R_i) + r_1(r_2 - r_1) + \cdots + r_{N-1}(r_{N-2} - r_{N-1}) + r_N(r_{N-2} - r_N) = K' \quad (1)$$

where K' is the sum of all the K_n's.

If the zone layout is such that the drive capacity is at an extremum, then the partial derivatives of Equation 1 with respect to each of the radii will be zero. We can use this fact to generate a set of linear equations in the unknown radii.

For example, we differentiate Equation 1 with respect to r_1 to get

$$\frac{\partial f}{\partial r_1} = R_i + r_2 - 2r_1 = 0$$

The derivative with respect to the j^{th} zone radius is

$$\frac{\partial f}{\partial r_j} = r_{j-1} + r_{j+1} - 2r_j = 0$$

The complete system of equations is

$$\begin{array}{rcl}
-2r_1 + r_2 & = & -R_i \\
r_1 - 2r_2 + r_3 & = & 0 \\
r_2 - 2r_3 + r_4 & = & 0 \\
\ddots & & \\
r_{N-3} - 2r_{N-2} + r_{N-1} & = & 0 \\
r_{N-2} - 2r_{N-1} & = & -R_o
\end{array}$$

which consists of $N-1$ equations in $N-1$ unknowns. This linear system is easily solved for the various r_j, but we want to derive the formulæ for the radii in terms of the given parameters. The following derivation completes the solution.

The system of equations derived above has a 'telescoping' property with respect to the variables from r_2 to r_{N-1} in the sense that a simple addition of some of the equations causes certain variables to drop out. For example, adding all the equations together results in

$$r_1 + r_{N-1} = R_i + R_o. \tag{2}$$

This can be restated as $r_1 - R_i = R_o - r_{N-1}$ which shows that the length of the inside zone is equal to the length of the outside zone. It is possible to show that the length of the second zone from the inside is equal to that of the second zone from the outside, etc. But we have a different use for the telescoping property.

We will develop a new system of equations from the old one as follows. Add the first equation to the second and replace the second with this sum. Similarly, add the new second equation to the third and replace the third with this sum. Continue until the full system is generated. This system is

$$\begin{array}{lllllll}
-2r_1 & + r_2 & & & & = - & R_i \\
-r_1 & - r_2 & + r_3 & & & = - & R_i \\
-r_1 & & - r_3 & + r_4 & & = - & R_i \\
 & & & \ddots & & & \\
-r_1 & & & & - r_{N-2} + r_{N-1} & = - & R_i \\
-r_1 & & & & - r_{N-1} & = - & R_i - R_o
\end{array}$$

where, on summing the equations, all variables except r_1 will drop out, leaving

$$N(-r_1) = (N-1)(-R_i) - R_o$$

which can be solved for r_1. The formula for r_1 is then

$$r_1 = \frac{(N-1)R_i + R_o}{N}.$$

An obvious way to solve for r_2 is to back substitute in the first equation of the system. Another way is to return to the original system and add the second equation to the third equation, replacing the third equation with this sum. Then replace the fourth with the sum of the new third and the old fourth, etc. The resulting system of equations can be added together producing an equation in r_1 and r_2 which, on substituting our previously determined value of r_1, gives r_2. Using either method the result is

$$r_2 = \frac{(N-2)R_i + 2R_o}{N}$$

Continuing in this manner, we find the equation for the j^{th} radius is

$$r_j = \frac{(N-j)R_i + jR_o}{N}$$

To prove that this represents zones of equal width across the disk, we form the difference between two adjacent zones

$$r_{j+1} - r_j = \frac{(N - (j+1))R_i + (j+1)R_o}{N} - \frac{(N-j)R_i + jR_o}{N}$$

which reduces to

$$r_{j+1} - r_j = \frac{R_o - R_i}{N}.$$

In other word, the width of each zone is simply a constant equal to the span of the recording area divided by the number of zones.