Engineering Notebook

miscellaneous problems and solutions

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Vol.1 no.14

Coefficients for Savitzky-Golay Filters

Savitzky and Golay[1] defined a family of filters which are suitable for smoothing and/or differentiating sampled data. The data are assumed to be taken at equal intervals.

The smoothing strategy is derived from the least squares fitting of a lower order polynomial to a number of consecutive points. For example, a cubic curve which is fit to 5 or more points in a least squares sense can be viewed as a smoothing function.

Their method consists of finding coefficients for the jth order smoothing polynomial in terms of the values of some number, $k > j+1$, of adjacent points and computing the value of the polynomial at the point to be smoothed.

At first glance, it appears that the computation of the appropriate coefficients for the cubic needs to be repeated for each point. However, by solving the appropriate equations in terms of a general point set it is possible to write an expression which is a weighted sum of neighboring points with weights constant for a given polynomial order and number of points.

We must solve the matrix equations:

$$\mathbf{Ax} = \mathbf{y}$$

where

$$\mathbf{A} = \begin{vmatrix} i_a^0 & i_a^1 & \cdots & i_a^n \\ i_b^0 & \ddots & & \vdots \\ \vdots & & & \vdots \\ i_q & \cdots & i_q^{n-1} & i_q^n \end{vmatrix}$$

[1] "Smoothing and Differentiation of Data by Simplified Least Squares Procedures",Abraham Savitzky and Marcel J.E. Golay, *Analytic Chemistry*, vol. 36, no. 8, July 1964, pp. 1627-1639.

and the i_k are the relative distances from the point we are smoothing to the y_k.

An example matrix formulation with the vector **x** representing the coefficient vector is:

$$\begin{vmatrix} -2^0 & -2^1 & -2^2 & -2^3 \\ -1^0 & -1^1 & -1^2 & -1^3 \\ 0^0 & 0^1 & 0^2 & 0^3 \\ 1^0 & 1^1 & 1^2 & 1^3 \\ 2^0 & 2^1 & 2^2 & 2^3 \end{vmatrix} \begin{vmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{vmatrix} = \begin{vmatrix} y_{-2} \\ y_{-1} \\ y_0 \\ y_1 \\ y_2 \end{vmatrix}$$

where a cubic, $a_0 + a_1 i + a_2 i^2 + a_3 i^3$, is to be fit to 5 consecutive points, $y_{-2}, y_{-1}, y_0, y_1, y_2$, so that the central point at y_0 occurs where $i = 0$. In other words, the horizontal axis is shifted so that the y values are evenly spaced around the origin. By this means the value of the cubic at the smoothing point is simply the value of the expression for a_0.

Note that the matrix **A** contains the term 0^0. It is understood that this is equal to unity, as is any other quantity raised to the zero power.

As it stands, $\mathbf{Ax} = \mathbf{y}$ is an overdetermined system and a least squares solution to the system is the desired result. Note that the solution values for the coefficients of the cubic will be given symbolically in terms of the y_i, since no numerical values for them have been specified yet.

Least squares problems of this sort are easily solved by forming the 'normal' equations for the system. That is, we solve the overdetermined system

$$\mathbf{Ax} = \mathbf{y}$$

where the matrix **A** has fewer columns than rows, **y** has the same number of rows as **A** and **x** has the same number of columns as **A**.

The solution is formed by first multiplying **A** and **y** by the transpose of **A**, $\mathbf{A}^T$, giving

$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{y}.$$

where $\mathbf{A}^T\mathbf{A}$ is now a square matrix.

Now multiply both sides by the inverse of $\mathbf{A}^T\mathbf{A}$, so that

$$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}.$$

The solution to the normal equations provides more information than was expected. Specifically, we found that the expression for the coefficient of a_0 is a weighted function of y which statisfies our requirement for a smoothing function. However, we may differentiate the cubic and evaluate the derivative at zero to obtain a point on the derivative of the smoothed curve as well. This derivative is simply the expression for the coefficient a_1 already obtained in solving the normal equations. Similarly, higher derivatives of the smoothed curve are available as the coefficients $a_2, a_3, \cdots$. Note, however, that these coefficient expressions must be multiplied by $0!, 1!, 2!, \cdots$, as appropriate. That is, for the polynomial, p

$$\left.\frac{d^n p}{di^n}\right|_{i=0} = n! a_i.$$

To return to the example problem, we solve

$$\left|\begin{array}{rrrr} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{array}\right| \left|\begin{array}{c} a_0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right| = \left|\begin{array}{c} y_{-2} \\ y_{-1} \\ y_0 \\ y_1 \\ y_2 \end{array}\right|$$

which gives

$$a_0 = \frac{-3y_{-2} + 12y_{-1} + 17y_0 + 12y_1 - 3y_2}{35} \quad (1)$$

$$a_1 = \frac{y_{-2} - 8y_{-1} + 8y_1 - 1y_2}{12} \quad (2)$$

$$a_2 = \frac{2y_{-2} - y_{-1} - 2y_0 - y_1 + 2y_2}{14} \quad (3)$$

$$a_3 = \frac{-y_{-2} + 2y_{-1} - 2y_1 + y_2}{12}. \quad (4)$$

To form the derivatives, the third and fourth of these equations must be multiplied by 2! and 3!, respectively, as mentioned previously.

If the symbolic form of the weighting function is not convenient, we may instead simply compute coefficient values by replacing the symbolic vector y with a unit vector. To compute the full set of coefficients for all derivatives (regarding the smoothing function as a zeroth order derivative), solve $\mathbf{Ax} = \mathbf{I}$, where $\mathbf{I}$ is the $j+1^{th}$ order identity matrix.

We discussed the solution of the normal equations for the case in which the y_n are equally distributed around the point $i = 0$. However, it is legitimate to have more points on one side of zero than there is on the other. For example, to start up the smoothing process we might want the first point to be computed from a weighted function of the first 5 points. In this case, the $\mathbf{A}$ matrix needs to be generated differently.