Problems and Solutions

in Mathematics, Physics and Applied Sciences

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Maximally Flat Polynomials

When is a polynomial 'maximally' flat? Consider the general polynomial

$$f(x) = a + bx + cx^2 + dx^3 + \cdots + nx^q. \tag{1}$$

If it is to be flat at $x = 0$, then $f'(0) = 0$, or

$$f'(x) = b + 2cx + 3dx^2 + \cdots + qnx^{q-1} = 0, \qquad x = 0.$$

From this, $b = 0$. Although the slope of all members of this modified family of polynomials is zero at zero, the curvature may be arbitrarily large for polynomials with order greater than unity. But we can make the curvature zero by setting the coefficient of the 2nd order term to zero; *i.e.* we differentiate twice and set $f''(0) = 0$. Hence, we set $c = 0$. Continuing in this manner we set set $d = 0$, $e = 0$, etc. up to, but not including, the coefficient of the highest power term in x. The resulting polynomial will have the least deviation from flatness at zero for its given order. It is *maximally* flat.

An n^{th} order polynomial which is maximally flat at $x = 0$ will necessarily have the form $f(x) = a + bx^n$. For maximally flat low pass filters, *i.e.* Butterworth, the normalized magnitude response is given by

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \omega^{2n}}}$$