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A New Integration Method Providing the Accuracy of Gauss-Legendre with Error Estimation Capability (rev.)

by C. Bond, ©2002

1 Background

One of the best methods for non-adaptive numerical integration of arbitrary functions is the Gauss-Legendre method. This method is based on the assumption that the required integral can be found with sufficient accuracy by integrating a high order approximating polynomial. Gauss-Legendre attempts to provide the highest accuracy for an estimate of the true integral by exactly integrating the polynomial of largest degree possible for a given number of functions calls. That is, given n function calls, what is the highest degree polynomial we can integrate exactly? If we are not restricted in locating the function points (n degrees of freedom), an appropriate set of weights (n degrees of freedom) will support the exact integration of any polynomial up to degree $2n-1$. It happens that the ideal function points (abscissae) are determined from the Legendre polynomials, hence the name. The weights can be determined by linear algebra or more advance methods.

An attractive property of Gauss-Legendre integration is that all the function points are inside the range of integration and do not include the end points. This means that singularities at one or both endpoints are unlikely to cause the integrator to fail.

A deficiency in the basic Gauss-Legendre method is that there is no indication of how closely the result actually approximates the correct answer. It is easy to devise pathological functions which defy numerical integrators, and we would like to have some assurance that a result is acceptable. Of course, two estimates of the integral can be made with two different orders of Gauss-

Legendre, but the number of function calls increases considerably, since two different orders have different abscissae and weights. In cases where the function calls are expensive, this may be undesirable.

One approach to the problem of error estimation is the method of Kronrod, commonly called Gauss-Kronrod integration. Kronrod provides a means to determine an error estimate by first computing a Gauss-Legendre result and, after inserting additional function points between the original points, integrating with a revised set of weights. This gives two estimates of the same integral. If the difference between the estimates is sufficiently small, the value of the Kronrod estimate is accepted. In practice, the simple difference between the two estimates is normally not used in favor of a more complex computed error estimate, such as is done in Quadpack. In any case, the cost of obtaining that estimate is to approximately double the number of required function calls.

An iterative method by Patterson, involving progressive extensions and commonly referred to as Gauss-Patterson rules, modifies Gauss-Kronrod in the following way: If the result of the initial Gauss-Kronrod integration does not appear to be sufficiently accurate, as determined by the error metric, function points are inserted again between those already used. Revised weights are used to perform another integration and a new error metric is computed. This process continues until the error metric is acceptable.

All these methods are implemented by constructing tables of abscissae and weights beforehand. The typical number of function points ranges from about 7 to about 60. For Gauss-Legendre integration, it is customary to provide a set of tables containing abscissae and weights for several different orders of integration. For Gauss-Kronrod there is usually a set of Gauss-Legendre tables with an associated Kronrod extension. For Gauss-Patterson, there is usually a single Gauss-Legendre table, a Gauss-Kronrod extension, and a series of Gauss-Patterson exensions up to some limit, say about 60 function points.

Integrators based on the above strategies have been used successfully within the scientific community as methods of choice for a wide variety of problems. Variations have been devised to extend them to handle periodic functions and singular integrals. In all, the existing body of work can manage most integration problems easily.

Note, however, that none of the extensions to Gauss-Legendre is capable of exactly integrating a polynomial of order $2n - 1$, where n is the number of function calls. Hence, the cost of providing an error estimate is an increase in the number of required function calls beyond that theoretically required to obtain a satisfactory estimate of the integral itself.

2 Gauss-Bond Integration

On examining these integration methods, it is clear that Kronrod and Patterson accept the penalty of increasing the number of function calls in order to provide an error metric. Hence their *best estimate* of the integral is obtained from a less than optimal calculation.

In this paper, the author proposes another approach. The method is best understood by comparing it with Gauss-Kronrod integration. Suppose a Gauss-Kronrod integrator consists of a 10th order Gauss-Legendre integrator; *i.e.* there are 10 abscissae inside the range of integration. The Kronrod extension places 11 additional abscissae by inserting new points between the existing Gauss-Legendre points plus 1 new point between each of the outer Gauss-Legendre points and the end points of the interval.

For the Gauss-Kronrod integration above, 21 function calls are required. The Gauss-Legendre portion exactly integrates any polynomial of order 21, and the extension should integrate a 32nd degree polynomial exactly. Hence, for 21 function calls we have an estimate of the integral which provides the accuracy of a 32nd degree polynomial, plus a means to estimate the error.

The downside of this is that the best estimate of the integral for the 21 function calls is equivalent to that of a Gauss-Legendre integration of order 17, which requires only 17 function calls. If we had begun with a Gauss-Legendre integration of order 17, we would have had an estimate of the integral of slightly better accuracy and 4 fewer function calls, but with no way of knowing how much confidence to place in the result.

Assuming that function calls are expensive, the challenge is to find a way to get a viable error estimate from a Gauss-Legendre integration *without additional function calls.*

One way to do this is to drop one or more of the Gauss-Legendre abscissae and integrate using a new set of weights on the remaining abscissae. In other words, an estimate of the integral is taken from the Gauss-Legendre integration of order, say, 17. The center abscissae is dropped and a new estimate is computed from these abscissae and a different set of weights for this 16th order integration. An error metric can then be computed from the difference between the two estimates. If it is sufficiently small, the result of the original integration is returned to the user. This integral should be exact for any polynomial of degree 33 or less. The reduced accuracy integral used for error estimation, however, is only capable of exactly integrating a polynomial of order 16. This disparity between the expected accuracies of the two estimates complicates the error estimation calculation, but can be managed by carefully constructing the error metric. Note, however, that if the two estimates of the integral agree closely, the Gauss-Legendre result is almost certainly accurate. It is only when the two results differ appreciably that our confidence suffers.

If the chosen Gauss-Legendre integrator is of even order, the inner two abscissae can be dropped and a center abscissae added to achieve the reduction by one symmetrically. This requires one additional function call.

The two most important properties of the chosen set of abscissae and weights for any integration method are:

1. the abscissae should be within the range of integration,
2. the weights should all be positive.

The first requirement is automatically met by any of the strategies discussed here. The second requirement assures that cancellation errors do not interfere with the estimate.

Fortunately, it is not necessary to labor over complex proofs that the *all-positive* requirement is met. It is sufficient to construct the tables of abscissae and weights and verify them by inspection.

For the case of Kronrod and Patterson rules, the all-positive weight requirement has been verified by construction of the tables. In this paper, the tables for Gauss-Bond integration are provided and, as required, the weights are all positive.

3 Tables of Abscissae and Weights

In this section, Gauss-Bond rules for orders 20,21,30,31,40,41,50,51,60 and 61 are tabulated to 60 decimal places.

3.1 Gauss-Bond Rules for Order 20

Gauss-Legendre Abscissae for order 20

$\pm$	0.993128599185094924786122388471320278222647130901655896148184
$\pm$	0.963971927277913791267666131197277221912060327806188856063538
$\pm$	0.912234428251325905867752441203298113049184797423691774795882
$\pm$	0.839116971822218823394529061701520685329629365065637373252493
$\pm$	0.746331906460150792614305070355641590310730679569176444139546
$\pm$	0.636053680726515025452836696226285936743389116799368463939447
$\pm$	0.510867001950827098004364050955250998425491329202426833472349
$\pm$	0.373706088715419560672548177024927237395746321705682711827949
$\pm$	0.227785851141645078080496195368574624743088937682927472314636
$\pm$	0.076526521133497333754640409398838211004796266813497500804795

Gauss-Legendre Weights for order 20

+	0.017614007139152118311861962351852816362143105543336732524349
+	0.040601429800386941331039952274932109879090639989951536817607
+	0.062672048334109063569506535187041606351601076578436364099584
+	0.083276741576704748724758143222046206100177828583163290744882

$+$ 0.101930119817240435036750135480349876166691656023392556261972
$+$ 0.118194531961518417312377377711382287005041219548968775446890
$+$ 0.131688638449176626898494499748163134916110511146983526996436
$+$ 0.142096109318382051329298325067164933034515413392020303337367
$+$ 0.149172986472603746787828737001969436692679904081368316496211
$+$ 0.152753387130725850698084331955097593491948645112378597274701

Gauss-Bond Abscissae for Order 19

$\pm$ 0.993128599185094924786122388471320278222647130901655896148184
$\pm$ 0.963971927277913791267666131197277221912060327806188856063538
$\pm$ 0.912234428251325905867752441203298113049184797423691774795882
$\pm$ 0.839116971822218823394529061701520685329629365065637373252493
$\pm$ 0.746331906460150792614305070355641590310730679569176444139546
$\pm$ 0.636053680726515025452836696226285936743389116799368463939447
$\pm$ 0.510867001950827098004364050955250998425491329202426833472349
$\pm$ 0.373706088715419560672548177024927237395746321705682711827949
$\pm$ 0.227785851141645078080496195368574624743088937682927472314636
0.000000000000000000000000000000000000000000000000000000000000

Gauss-Bond Weights for order 19

$+$ 0.018083140926107477089728025787776949704560773447753237282771
$+$ 0.038933502007589214227204443766969602035198803885557723459891
$+$ 0.066044501758134835108630739562378141385132825886403071919000
$+$ 0.077665202857597121231849652605330616705549327938153262080472
$+$ 0.110470916641747572036594953921933654768325968087015183105418
$+$ 0.105683153028792177500720169698650169832553290507545719346892
$+$ 0.150006539820487719021423003163552281998949173585316980807282
$+$ 0.114030159720729639536647331012614128859633494938137825666901
$+$ 0.198698769670307169891983507498327006435351279928495246116046
$+$ 0.240768227137014148710436345964934896549490123591243500430652

3.2 Gauss-Bond Rules for Order 21

Gauss-Legendre Abscissae for order 21

± 0.993752170620389500260242035937940929193338454782329185580874
± 0.967226838566306294316622214907695161424693687329846849952971
± 0.920099334150400828790187133714968894159147609648221697176135
± 0.853363364583317283647250638587567670276105803179343970742992
± 0.768439963475677908615877851306228034820976705771369550868727
± 0.667138804197412319305966669990339162597029343311402842475467
± 0.551618835887219807059018796724313286622060224230679665004306
± 0.424342120207438783573668888543788052096445231839634584224258
± 0.288021316802401096600792516064600319909018263646033228754282
± 0.145561854160895090937030982338686330116326024437937757421488
0.000000000000000000000000000000000000000000000000000000000000

Gauss-Legendre Weights for order 21

\+ 0.016017228257774333324224616858471015265890422117902482548178
\+ 0.036953789770852493799950668299329666188944308148729837427169
\+ 0.057134425426857208283635826472447957491287596826256041442507
\+ 0.076100113628379302017051653300183179226154352423966028521468
\+ 0.093444423456033861553289741113932088483526647361592844866695
\+ 0.108797299167148377663474578070105642033699595793856330552782
\+ 0.121831416053728534195367177125733598356337625561537450242984
\+ 0.132268938633337461781052574496775604329011540143156823218350
\+ 0.139887394791073154722133423867583110892793160954919604373731
\+ 0.144524403989970059063827166553752543609949962197099834752473
\+ 0.146081133649690427191985147683371188244809576941965444107323

Gauss-Bond Abscissae for Order 20

± 0.993752170620389500260242035937940929193338454782329185580874

± 0.9672268385663062943166222149076951614246936873298468499529 71

± 0.9200993341504008287901871337149688941591476096482216971761 35

± 0.8533633645833172836472506385875676702761058031793439707429 92

± 0.7684399634756779086158778513062280348209767057713695508687 27

± 0.6671388041974123193059666699903391625970293433114028424754 67

± 0.5516188358872198070590187967243132866220602242306796650043 06

± 0.4243421202074387835736688885437880520964452318396345842242 58

± 0.2880213168024010966007925160646003199090182636460332287542 82

± 0.1455618541608950909370309823386863301163260244379377574214 88

Gauss-Bond Weights for order 20

+ 0.0106185068786295819657729377488682681187033165689997606562 34

+ 0.0556095497747986766748165695362731962206005172703707466070 47

+ 0.0213509066165672182893351191358895789307621254191057633322 03

+ 0.1310657484884345603536942231235444263643551892298280313452 83

+ 0.0186790125246841257874795642403611013252218427303486145809 81

+ 0.2027098392220866781418041863142959851456590502880650438120 97

+ 0.0105576177287917472193085760120418324489471000641057071812 69

+ 0.2581369541881641159744305533486987191881475634245496445940 01

+ 0.0029943519231780123242311762320411344509993554935856342606 52

+ 0.2882775126546652832691270943079857578066039395110410536302 32

3.3 Gauss-Bond Rules for Order 30

Gauss-Legendre Abscissae for order 30

± 0.9968934840746495402716300509186952833408820381177507901080 90

± 0.9836681232797472099700325816056628019403178547097113635171 57

± 0.9600218649683075122168710255817976629303592174039233994856 06

± 0.9262000474292743258793242770804740040864745368253290609109 36

± 0.8825605357920526815431164625302255900566891471464842320681 70

± 0.8295657623827683974428981197325019164390686961703416788067 36

$\pm$ 0.767777432104826194917977340974503131694883617232908453206216

$\pm$ 0.697850494793315796932292388026640068382353800653954656379389

$\pm$ 0.620526182989242861140477556431189299207364692829528132594675

$\pm$ 0.536624148142019899264169793311072794164178006930297105452346

$\pm$ 0.447033769538089176780609900322854000162407593861424409754088

$\pm$ 0.352704725530878113471037207089373860653631008021425626593835

$\pm$ 0.254636926167889846439805129817805107882789303302518426164009

$\pm$ 0.153869913608583546963794672743255920418551971244338461718785

$\pm$ 0.051471842555317695833025213166722573749141453666569564255100

Gauss-Legendre Weights for order 30

$+$ 0.007968192496166605615465883474673622450480696587151721229497

$+$ 0.018466468311090959142302131912047269096206533968181403371324

$+$ 0.028784707883323369349719179611292043639588894546287496474219

$+$ 0.038799192569627049596801936446347692033200976766395352107783

$+$ 0.048402672830594052902938140422807517815271809197372736345250

$+$ 0.057493156217619066481721689402056128797120670721763134548778

$+$ 0.065974229882180495128128515115962361237442953656660378967091

$+$ 0.073755974737705206268243850022190734153770526037049438941320

$+$ 0.080755895229420215354694938460529730875892803708439299890292

$+$ 0.086899787201082979802387530715125702576753328743545344012230

$+$ 0.092122522237786128717632707087618767196913234418234107527651

$+$ 0.096368737174644259639468626351809865096406461430160245912938

$+$ 0.099593420586795267062780282103569476529869263666704277221278

$+$ 0.101762389748405504596428952168554044632706289487126840864151

$+$ 0.102852652893558840341285636705415043868375557064928222586197

Gauss-Bond Abscissae for Order 29

$\pm$ 0.996893484074649540271630050918695283340882038117750790108090

$\pm$ 0.983668123279747209970032581605662801940317854709711363517157

$\pm$ 0.960021864968307512216871025581797662930359217403923399485606

$\pm$ 0.926200047429274325879324277080474004086474536825329060910936

$\pm$ 0.882560535792052681543116462530225590056689147146484232068170
$\pm$ 0.829565762382768397442898119732501916439068696170341678806736
$\pm$ 0.767777432104826194917977340974503131694883617232908453206216
$\pm$ 0.697850494793315796932292388026640068382353800653954656379389
$\pm$ 0.620526182989242861140477556431189299207364692829528132594675
$\pm$ 0.536624148142019899264169793311072794164178006930297105452346
$\pm$ 0.447033769538089176780609900322854000162407593861424409754088
$\pm$ 0.352704725530878113471037207089373860653631008021425626593835
$\pm$ 0.254636926167889846439805129817805107882789303302518426164009
$\pm$ 0.153869913608583546963794672743255920418551971244338461718785
0.000000000000000000000000000000000000000000000000000000000000

Gauss-Bond Weights for order 29

$+$ 0.007851619241992019546663473957436218739216626949182709577875
$+$ 0.018877475487657606073123255428845429215686106569734867330743
$+$ 0.027967005923089950825048471808542696884482254506906556503979
$+$ 0.040124567671169544959567871659779798605862094950970283129200
$+$ 0.046465246486082525749748834903483796517147242582818856314482
$+$ 0.060161017325200486982480524950353728123760121309864612165742
$+$ 0.062431228243307932139620935139942826291238530832802495543759
$+$ 0.078363316260585708470837783141982725098517053475866950410961
$+$ 0.074819819025395219572808337581227971746273611315567672678794
$+$ 0.094561805710250302285506062972064606677557366910881217576335
$+$ 0.082083654248658295053381915567118579306651042871037788409185
$+$ 0.109982025855300676701466454035682116488187486553346698012914
$+$ 0.079783148637564188269883583833443696334169078433718726513385
$+$ 0.135622663041811716008704227160882988597036856295663976768554
$+$ 0.161810813683867654722316535718425642748429052883273178128186

3.4 Gauss-Bond Rules for Order 31

Gauss-Legendre Abscissae for order 31

$\pm$ 0.997087481819477074055626554223102508251445334117069630792120
$\pm$ 0.984685909665152484002465166734684710938518269514643670945710
$\pm$ 0.962503925092949661789052404105858301897219485152558376773305
$\pm$ 0.930756997896648164956945759729263512813397687258082036455254
$\pm$ 0.889760029948271043374192008982159261543597411865190949763006
$\pm$ 0.839920320146267340086904535940178356054737301074749522831187
$\pm$ 0.781733148416624940406360020194684491095410372013398880064652
$\pm$ 0.715776784586853283905970865366485955890193476169588432627838
$\pm$ 0.642706722924260346184418203232501453991690683336206756508993
$\pm$ 0.563249161407149262720944923595161427948602273166887030985425
$\pm$ 0.478193782044902480440594039356485748439523371497423124837801
$\pm$ 0.388385901608232943061351461287520101985106267893544035757459
$\pm$ 0.294718069981701616617903897671704338752855185278358556935979
$\pm$ 0.198121199335570628772412996032833937774794812161554879986560
$\pm$ 0.099555312152341520325174790118940733838995358777527306060095
0.000000000000000000000000000000000000000000000000000000000000

Gauss-Legendre Weights for order 31

$+$ 0.007470831579248775858696875032212407022150852484048039699280
$+$ 0.017318620790310582463157996086821141366272862767995776632777
$+$ 0.027009019184979421800608708091932156142356646002116645856907
$+$ 0.036432273912385464024392010467803442327145332047667685901459
$+$ 0.045493707527201102902315857894629677263552719310028964893563
$+$ 0.054103082424916853711666259086620042720389462596211807694057
$+$ 0.062174786561028426910343543686696667158728799364651315281556
$+$ 0.069628583235410366167756126255125710515085707397343393711118
$+$ 0.076390386598776616426357674901330644925756913694208654092227
$+$ 0.082392991761589263903823367431961770615768266556646277810409

$+$ 0.087576740608477876126198069695333092229258159771204251503548

$+$ 0.091890113893641478215362871607150125497310825379157804178465

$+$ 0.095290242912319512807204197487596684541324738246959022711394

$+$ 0.097743335386328725093474010978996703835728788676095114509421

$+$ 0.099225011226672307874875514428615014017543219955563069781079

$+$ 0.099720544793426451427533833734349439643253411500204351485479

Gauss-Bond Abscissae for Order 30

$\pm$ 0.997087481819477074055626554223102508251445334117069630792120

$\pm$ 0.984685909665152484002465166734684710938518269514643670945710

$\pm$ 0.962503925092949661789052404105858301897219485152558376773305

$\pm$ 0.930756997896648164956945759729263512813397687258082036455254

$\pm$ 0.889760029948271043374192008982159261543597411865190949763006

$\pm$ 0.839920320146267340086904535940178356054737301074749522831187

$\pm$ 0.781733148416624940406360020194684491095410372013398880064652

$\pm$ 0.715776784586853283905970865366485955589019347616958843262838

$\pm$ 0.642706722924260346184418203232501453991690683336206756508993

$\pm$ 0.563249161407149262720944923595161427948602273166887030985425

$\pm$ 0.478193782044902480440594039356485748439523371497423124837801

$\pm$ 0.388385901608232943061351461287520101985106267893544035757459

$\pm$ 0.294718069981701616617903897671704338752855185278358556935979

$\pm$ 0.198121199335570628772412996032833937774794812161554879986560

$\pm$ 0.099555312152341520325174790118940733838995358777527306060095

Gauss-Bond Weights for order 30

$+$ 0.009552490928608382223880171665552889977093183660011941649447

$+$ 0.010073586117504347601803303809047051072308671093648748767033

$+$ 0.041087146842768671378078083194309659847501732332963563592093

$+$ 0.014393510957778929506842716522505552664714800540583816154141

$+$ 0.076236326306895907489998276557343087589292022167987585409714

$+$ 0.014239994535372642485312301714489674576865671077713346860613

$+$ 0.111278540487992827870250392911490785446758074186342452648292

+ 0.011438863123891383559732300130247762583306191309414143429793
+ 0.143256197333213385029745581681894635413888318696292852357253
+ 0.007495300001137644677846974675689398048023196341211761262546
+ 0.169651009099583147123764621424676167662249590880964251170685
+ 0.003679558288079554795370743217761734514974726199929547861181
+ 0.188440813915275118708151399873779519201282057274220737499117
+ 0.000973357683569504896085793736842243267471474332005960342156
+ 0.198203304378328552653137338884369838134270289906709290995939

3.5 Gauss-Bond Rules for Order 40

Gauss-Legendre Abscissae for order 40

± 0.998237709710559200349622702420586492335770381595045808577581
± 0.990726238699457006453054352221372154962222081351086024878352
± 0.977259949983774262663370283712903806978667932037984851175804
± 0.957916819213791655804540999452759285094883490602744761591148
± 0.932812808278676533360852166845205716434753575282688898929952
± 0.902098806968874296728253330868493103584488081057664431112536
± 0.865959503212259503820781808354619963570546553011094983606217
± 0.824612230833311663196320230666098773907240384242979438623162
± 0.778305651426519387694971545506494848020691316126881762542263
± 0.727318255189927103280996451754930548557378673533316562403522
± 0.671956684614179548379354514961494109970325981383838269965139
± 0.612553889667980237952612450230694877380123781683135778757367
± 0.549467125095128202075931305529517970233975101595637141746493
± 0.483075801686178712908566574244823004599022395533099841136162
± 0.413779204371605001524879745803713682974099624052904661350012
± 0.341994090825758473007492481179194310066953620027313547235050
± 0.268152185007253681141184344808596183424804373236236683321946
± 0.192697580701371099715516852065149894814092021105201079079604

$\pm$ 0.116084070675255208483451284408024113768728530854211087557655

$\pm$ 0.038772417506050821933193444024623294679364634383139947198477

Gauss-Legendre Weights for order 40

$+$ 0.004521277098533191258471732878185332727831110199706241869181

$+$ 0.010498284531152813614742171067279652376792621315797356467534

$+$ 0.016421058381907888712863484882363927292342293346958645582974

$+$ 0.022245849194166957261504324184208573207033196679355587584551

$+$ 0.027937006980023401098489157507721077302550862050767791132672

$+$ 0.033460195282547847392678183086410848977241786653765919852723

$+$ 0.038782167974472017639972031290446162253459211232027534050595

$+$ 0.043870908185673271991674686041715495811006837170238588858361

$+$ 0.048695807635072232061434160448146388067843027377121400152438

$+$ 0.053227846983936824354996479772260504555321171822007893991711

$+$ 0.057439769099391551366617730910425985600104835854454774028546

$+$ 0.061306242492928939166537996408398595902593763511175060695761

$+$ 0.064804013456601038074554529566752730032692964208489133544205

$+$ 0.067912045815233903825690108231923985984197238379285589516653

$+$ 0.070611647391286779695483630855286832359559103995585092649872

$+$ 0.072886582395804059060510683442517835857559080985796983255344

$+$ 0.074723169057968264200189336261324673191202934420357578847714

$+$ 0.076110361900626242371558075922494823012559553845068365314109

$+$ 0.077039818164247965588307534283810248524439754163937314935990

$+$ 0.077505947978424811263723962958326326963668652788103147669063

Gauss-Bond Abscissae for Order 39

$\pm$ 0.998237709710559200349622702420586492335770381595045808577581

$\pm$ 0.990726238699457006453054352221372154962222081351086024878352

$\pm$ 0.977259949983774262663370283712903806978667932037984851175804

$\pm$ 0.957916819213791655804540999452759285094883490602744761591148

$\pm$ 0.93281280827867653336085216684520571643475357528268898929952

$\pm$ 0.902098806968874296728253330868493103584488081057664431112536

$\pm$ 0.865959503212259503820781808354619963570546553011094983606217
$\pm$ 0.824612230833311663196320230666098773907240384242979438623162
$\pm$ 0.778305651426519387694971545506494848020691316126881762542263
$\pm$ 0.727318255189927103280996451754930548557378673533316562403522
$\pm$ 0.671956684614179548379354514961494109970325981383838269965139
$\pm$ 0.612553889667980237952612450230694877380123781683135778757367
$\pm$ 0.549467125095128202075931305529517970233975101595637141746493
$\pm$ 0.483075801686178712908566574244823004599022395533099841136162
$\pm$ 0.413779204371605001524879745803713682974099624052904661350012
$\pm$ 0.341994090825758473007492481179194310066953620027313547235050
$\pm$ 0.268152185007253681141184344808596183424804373236236683321946
$\pm$ 0.192697580701371099715516852065149894814092021105201079079604
$\pm$ 0.116084070675255208483451284408024113768728530854211087557655
0.000000000000000000000000000000000000000000000000000000000000

Gauss-Bond Weights for order 39

+ 0.004564456435554324525901314270762114877046502953249960272276
+ 0.010346489934870609549990901009466228094001104979047390161408
+ 0.016721413574368883171368257769278375596676026034114403817403
+ 0.021763051515650997800116709820307736559503032844217548650620
+ 0.028634517335065512569144363054965306833160144513270600477169
+ 0.032514964762533129758708920910356478998964613241599374723155
+ 0.040010751423375528102621040127906475328542661250723697467212
+ 0.042318733617604129020097907153496343666672015306740068561140
+ 0.050618869244765248073949329929717747389328983424448320256696
+ 0.050876177653759185784199941692333860303409180905601867028759
+ 0.060293153294160993500265150895164933743550006216851529517718
+ 0.057854901270542614319745519426008499013317776422609340542922
+ 0.068985500396567399328071991002566459252368717615661743152343
+ 0.062809695071441891101942698367793507036586200380212203835163
+ 0.076927151130872075139349226756683687362471853653347918976009
+ 0.064873262690825862295101386234733507119871767118617208553219

$+\quad 0.08533180286437347402794847959832546350731326078713265840728$2

$+\quad 0.06093475712609799377601018282860900913820502462565832050064$5

$+\quad 0.10269387252849249560039248356686321384632777495943726640196$0

$+\quad 0.12185295625815530511014839116932210466536670553491715739380$3

3.6 Gauss-Bond Rules for Order 41

Gauss-Legendre Abscissae for order 41

$\pm\quad 0.998321588574771441519188508088377567931504351513670816724875$

$\pm\quad 0.991167109699016308250158894573169657692756013570006455275276$

$\pm\quad 0.978338673561083384469170706673967198375636781918662575825969$

$\pm\quad 0.959906891730346226099441696068686582066951090829481525777446$

$\pm\quad 0.935976987497853825682318275284766266271237118087368629786108$

$\pm\quad 0.906685944758101172958340413230722973354307185322318918496168$

$\pm\quad 0.872201511692441408833670574234329460282042655539578453974423$

$\pm\quad 0.832721200401361331244272779048543799976975992978639714608306$

$\pm\quad 0.788471145047409372736221781466770288994712079098797127646762$

$\pm\quad 0.739704803069926181060168744468559000395254439052650790713605$

$\pm\quad 0.686701502034951289584603856529134812066438460149765702244645$

$\pm\quad 0.629764839072196320488649091695103501078514780739572223782627$

$\pm\quad 0.569220941610215869654747215282980251610456923979652542519820$

$\pm\quad 0.505416599199406032708336070202694764951075546015289827442265$

$\pm\quad 0.438717277051407088517119856728039248421785758759576111089441$

$\pm\quad 0.369505022640481441428366913299352056240869713319765195396400$

$\pm\quad 0.298176277341824865922982880478924304139955868446236839278872$

$\pm\quad 0.225139605633422775605786256655472330299912420644314536993054$

$\pm\quad 0.150813354863992163574377967884582375775630945329547962435291$

$\pm\quad 0.075623258989162996923766193651200267481050635606682826931774$

$\quad 0.000000000000000000000000000000000000000000000000000000000000$

Gauss-Legendre Weights for order 41

$+$ 0.004306140358164887684004477904654486186362203995300038049301
$+$ 0.009999938773905945338496296629698839550224935771146367843521
$+$ 0.015644938407818588530826844479533677440817889369640445451754
$+$ 0.021201063368779553075697033493376626571717054327350488575703
$+$ 0.026635899207110445467548575258713277044690492005238642233661
$+$ 0.031918211731699281787066946857145543701187753297500148770319
$+$ 0.037017716703507988435261251580160668606271546321855703354496
$+$ 0.041905195195909689429340274311048837936266377432010053300525
$+$ 0.046552648369014342060756586864611423130990198580577948884665
$+$ 0.050933454294617494781170357115688631420803446786858871196009
$+$ 0.055022519242578741880146810171422269973775336605359687570194
$+$ 0.058796420949871944991185853380734274000881238602719434573657
$+$ 0.062233542580966316471573417083346485430159869944939179270454
$+$ 0.065314196453527410436163712653796340419859236721067934032855
$+$ 0.068020736760876766735533239726248928256699120138598673747095
$+$ 0.070337660620817497481658989969881456001233380167819858157158
$+$ 0.072251696861023073396346398348787354885294770524887188056078
$+$ 0.073751882027223469939280818324848679477916280428358885834691
$+$ 0.074829623176221551891305072633505322945683067378053653415068
$+$ 0.075478747092715824027247062674616749073606360903088103519264
$+$ 0.075695535647298372318779961076360255891118881395257388327065

Gauss-Bond Abscissae for Order 40

$\pm$ 0.998321588574771441519188508088377567931504351513670816724875
$\pm$ 0.991167109699016308250158894573169657692756013570006455275276
$\pm$ 0.978338673561083384469170706673967198375636781918662575825969
$\pm$ 0.959906891730346226099441696068686582066951090829481525777446
$\pm$ 0.935976987497853825682318275284766266271237118087368629786108
$\pm$ 0.906685944758101172958340413230722973354307185322318918496168
$\pm$ 0.872201511692441408833670574234329460282042655539578453974423

$\pm$ 0.832721200401361331244272779048543799976975992978639714608306
$\pm$ 0.788471145047409372736221781466770288994712079098797127646762
$\pm$ 0.739704803069926181060168744468559000395254439052650790713605
$\pm$ 0.686701502034951289584603856529134812066438460149765702244645
$\pm$ 0.629764839072196320488649091695103501078514780739572223782627
$\pm$ 0.569220941610215869654747215282980251610456923979652542519820
$\pm$ 0.505416599199406032708336070202694764951075546015289827442265
$\pm$ 0.438717277051407088517119856728039248421785758759576111089441
$\pm$ 0.369505022640481441428366913299935205624086971331976519539640
$\pm$ 0.298176277341824865922982880478924304139955868446236839278872
$\pm$ 0.225139605633422775605786256655472330299912420644314536993054
$\pm$ 0.150813354863992163574377967884582375775630945329547962435291
$\pm$ 0.075623258989162996923766193651200267481050635606682826931774

Gauss-Bond Weights for order 40

$+$ 0.003260551064259201718773772539287495705617276528288156095977
$+$ 0.013648646507042018550403481366924774664458118494811730555497
$+$ 0.008521086113917623582887469805262474200880973283694770923786
$+$ 0.032430721595149749343797373010178580552391935483877681105028
$+$ 0.010827533920891512197481290709975879680577984695527174295794
$+$ 0.052651468358141811583354051299371136013800190201673818543050
$+$ 0.011124911674575075376027106954357802370967913892437684799146
$+$ 0.073089618766908413892476131162441784230789248803273108463881
$+$ 0.010040859802331973747905700977510766635325438955566622365153
$+$ 0.092717218034650925534933613493894624791863732980367815620077
$+$ 0.008108392048692842176506163783114198218907260726019715281138
$+$ 0.110618147822051369944313537409703531243199511673025519347276
$+$ 0.005802674131928335290860067008979503755038511179602939288636
$+$ 0.125986019657828299582495790377713406767111140763996694641517
$+$ 0.003539350492498248041510165526430752864236743242184730853319
$+$ 0.138141099874817992415363255925850811787735319470238284568133
$+$ 0.001662211586169685408213286422138507048815812508685778877461

$+$ 0.146551015241357079830977822843310358579806446864675821961700
$+$ 0.000429109037119896629344881396294985674356787318946480937821
$+$ 0.150849364269667945152375037987258625214119652933105471475609

3.7 Gauss-Bond Rules for Order 50

Gauss-Legendre Abscissae for order 50

$\pm$ 0.998866404420071050185459444974218505996243508797651920504471
$\pm$ 0.994031969432090712585108200420694728157477949508108066029581
$\pm$ 0.985354084048005882309009625632489404015592578297745647819961
$\pm$ 0.972864385106692073713344104606252053669173310218896917060272
$\pm$ 0.956610955242807942997745644156622094051433971318970127448918
$\pm$ 0.936656618944877933780874947272496602153731377101620502603874
$\pm$ 0.913078556655791893089735642771657094784187893191426469463937
$\pm$ 0.885967979523613048637540982466753634194289926624633422655007
$\pm$ 0.855429769429946084611362643934757467654832563693665946545980
$\pm$ 0.821582070859335948356254110873939537760740812316739864983203
$\pm$ 0.784555832900399263905305196340991200847315605607463638913445
$\pm$ 0.744494302226068538260536252682194242870187171084062298272620
$\pm$ 0.701552468706822251089546257883655728149718381308820822110192
$\pm$ 0.655896465685439360781624864003679819041409604770316597203754
$\pm$ 0.607702927184950239180381796391832893604204035550796533112874
$\pm$ 0.557158304514650054315522909625801607815897355099961357067039
$\pm$ 0.504458144907464201651459131849141192635377633115667990611522
$\pm$ 0.449806334974038789147131467778375817315063478381195286376743
$\pm$ 0.393414311897565127394229253823817270246138475040149243013642
$\pm$ 0.335500245419437356836988257291071697841217674278793578085711
$\pm$ 0.276288193779531990327645278521130185714800777021237491866890
$\pm$ 0.216007236876041756847284532617101333705755303116531185165476
$\pm$ 0.154890589998145902071628620941109501201849719504309887199543

$\pm$ 0.093174701560086140854450377639600347885671074212005062135525
$\pm$ 0.031098338327188876112328989665949194247296118116525856119042

Gauss-Legendre Weights for order 50

$+$ 0.002908622553155140958400724342855480806673006989675401940417
$+$ 0.006759799195745401502778878177985031801873856744388099690190
$+$ 0.010590548383650969263569681499241022339401856833491328836429
$+$ 0.014380822761485574419378908927324349937031836656738135914287
$+$ 0.018115560713489390351259943422354619844667379335659582455328
$+$ 0.021780243170124792981592069062690341227313535214787010502704
$+$ 0.025360673570012390440194878385442723460161341859218754670467
$+$ 0.028842993580535198029906373113232432517846954614386314693159
$+$ 0.032213728223578016648165827323003953448589152726186023677207
$+$ 0.035459835615146154160734611000975797096960096591052261501114
$+$ 0.038568756612587675244770150236385934864771800198452858472996
$+$ 0.041528463090147697422411978964067017808978066256968893881033
$+$ 0.044327504338803275492022286830394197460761380755620922341827
$+$ 0.046955051303948432965633013634987682514064375917548646921846
$+$ 0.049400938449466314921243580751432728692287103494701641353833
$+$ 0.051655703069581138489905295840095279649825480138071825167188
$+$ 0.053710621888996246523458797255664552768023218132871082106170
$+$ 0.055557744806212517623567425612269497595135274682575618214080
$+$ 0.057189925647728383723029315065993163011575314307463643414028
$+$ 0.058600849813222445835122436630848466209767421580149371646085
$+$ 0.059785058704265457509576405312585230796665916922528072190222
$+$ 0.060737970841770216031750015384811001609799117609432629612346
$+$ 0.061455899590316663756406786083915375097267394731136196227977
$+$ 0.061936067420683243384087509780830688572876857382015397022128
$+$ 0.062176616655347262321033107360613430867682260324880287546937

Gauss-Bond Abscissae for Order 49

$\pm$ 0.998866404420071050185459444974218505996243508797651920504471

± 0.99403196943209071258510820042069472815747794950810806602958 1

± 0.98535408404800588230900962563248940401559257829774564781996 1

± 0.97286438510669207371334410460625205366917331021889691706027 2

± 0.95661095524280794299774564415662209405143397131897012744891 8

± 0.93665661894487793378087494727249660215373137710162050260387 4

± 0.91307855665579189308973564277165709478418789319142646946393 7

± 0.88596797952361304863754098246675363419428992662463342265500 7

± 0.85542976942994608461136264393475746765483256369366594654598 0

± 0.82158207085933594835625411087393953776074081231673986498320 3

± 0.78455583290039926390530519634099120084731560560746363891344 5

± 0.74449430222606853826053625268219424287018717108406229827262 0

± 0.70155246870682225108954625788365572814971838130882082211019 2

± 0.65589646568543936078162486400367981904140960477031659720375 4

± 0.60770292718495023918038179639183289360420403555079653311287 4

± 0.55715830451465005431552290962580160781589735509996135706703 9

± 0.50445814490746420165145913184914119263537763311566799061152 2

± 0.44980633497403878914713146777837581731506347838119528637674 3

± 0.39341431189756512739422925382381727024613847504014924301364 2

± 0.33550024541943735683698825729107169784121767427879357808571 1

± 0.27628819377953199032764527852113018571480077702123749186689 0

± 0.21600723687604175684728453261710133370575530311653118516547 6

± 0.15489058999814590207162862094110950120184971950430988719954 3

± 0.09317470156008614085445037763960034788567107421200506213552 5

0.00000000000000000000000000000000000000000000000000000000000 0

Gauss-Bond Weights for order 49

\+ 0.00288868287855036865467401440334484398506258045364777306594 9

\+ 0.00682980095531852937665441922975298941766395582631293673318 3

\+ 0.01045238117230746630566904549926724211290874094978653975793 5

\+ 0.01460209088117085588480687955291263045568342234226464387156 1

\+ 0.01779751198881956116412395783769577215406105078490118450895 4

\+ 0.02220838604569726237909000671525516922859278144647995227832 0

$+ \quad 0.0248088973899854284238640901731785384500198917893677623954 96$
$+ \quad 0.029532667083943042857756867153459048462769456171660557279918$
$+ \quad 0.031370667871330883934260028680562078556387788616067620367103$
$+ \quad 0.036473564585020755163332409042953149026916804342139968857352$
$+ \quad 0.037364588182079982883875294698984791604900184298091027234370$
$+ \quad 0.042946238555722157254372926942074134732324547653704209090861$
$+ \quad 0.042668322502348799890089175430214918981848300337322875569815$
$+ \quad 0.048889814683148958632184600841118819881723340988129435121738$
$+ \quad 0.047147490455760362262246412734347566012949768715174450615090$
$+ \quad 0.054283743239750822355765688156016576818290652443058603744851$
$+ \quad 0.050633136417471063090330669211587930120430707927754759264185$
$+ \quad 0.059188695565166371263106950943477273770845685614575238728640$
$+ \quad 0.052854251275079675667401629235580894814758702271004274604030$
$+ \quad 0.063874233200171446403564818051184899937068546470747123978272$
$+ \quad 0.053186455087277350262491951359902304940516915974993167271912$
$+ \quad 0.069380617897253741833987642996515680558753067086372198646959$
$+ \quad 0.049188726888563978560243255010100308544903357708082174043436$
$+ \quad 0.082568063012692314832968270869015008406752220901390835712146$
$+ \quad 0.097721944370737641326277990462994858047735057773941374515848$

3.8 Gauss-Bond Rules for Order 51

Gauss-Legendre Abscissae for order 51

$\pm \quad 0.998909990848903495168995877273385637084249938575111612977051$
$\pm \quad 0.994261260436752574621084897949263128231784477968454277278541$
$\pm \quad 0.985915991735902996583885707558309632652115992556224484566752$
$\pm \quad 0.973903368019323867231755486394186639013106901689554435137002$
$\pm \quad 0.958267848613908194557707038316323727612444646454842233655520$
$\pm \quad 0.939067544002962383435367806390905144165164730768267936526322$
$\pm \quad 0.916373862309780230823571294251475879366043904795797069588505$

$\pm$ 0.890271218029527303277795370736726893557801394449321043531130
$\pm$ 0.860856711182292371473495743716111323577346914638034911360632
$\pm$ 0.828239763823064832854818424016562364262390186679011920334688
$\pm$ 0.792541712099381205234410878375875730157325296801126404034282
$\pm$ 0.753895354485375525763960025452473987630640624374368957642043
$\pm$ 0.712444457577036644580524855400145643036948463177564539567964
$\pm$ 0.668343221175370086864460419403988875929612565078091525009347
$\pm$ 0.621755704600723273755042745403315950054391949288853476007318
$\pm$ 0.572855216351303836522394702590190907420947759960678022943583
$\pm$ 0.521823669366185842514087784826818024569355047379422926201926
$\pm$ 0.468850904286041063610457258811622468806727482427668738182181
$\pm$ 0.414133983226303877936871809744657907923981925685998693125257
$\pm$ 0.357876456688409509775201088519666356896388431823667655371603
$\pm$ 0.300287606335331939530245649644420333331342481297453907665457
$\pm$ 0.241581666447798703846733114869262423738796774456755816882615
$\pm$ 0.181977026957077545323998701169214449923922918920115452813381
$\pm$ 0.121695421018888766963820420963181108745305234967309681175283
$\pm$ 0.060961100150578724734194706843205404697031462796017398755700
0.000000000000000000000000000000000000000000000000000000000000

Gauss-Legendre Weights for order 51

$+$ 0.002796807171089895575544216881809048224217656836477127598250
$+$ 0.006500337783252600292109375198537528833094843910372232719600
$+$ 0.010185191297821729939237591786606728385811481896556908302945
$+$ 0.013832634006477822296688453021856415956018344870720739787367
$+$ 0.017428714723401052259503646433292435568356521018264614037984
$+$ 0.020959988401703210579792618401514172073040181767462835441334
$+$ 0.024413300573781434273141565416898208743682825940503620005902
$+$ 0.027775798594162477195995665632208916368350869320276480152698
$+$ 0.031034971290160008454425502956547377296108571830158099091038
$+$ 0.034178693204188336236209338594767529048825439721341688612528
$+$ 0.037195268923260292842908275811871025134112280386813934789884

$+$ 0.040073476285496453186809115921395911201163908442291312651929
$+$ 0.042802607997880086653609514244285700837174874633008368662774
$+$ 0.045372511407650068748166814988437851313829598620529455302373
$+$ 0.047773626240623101999995353707353617340538983869091831080568
$+$ 0.049997020150057409779548855362005785568736536084059689175669
$+$ 0.052034421936697087564136447468662373215483867205893456075251
$+$ 0.053878252313045561434099301696971983760059987726189609017324
$+$ 0.055521652095738693016737059093624160780863871776739145187936
$+$ 0.056958507720258662100077726734277164878409776611670828974151
$+$ 0.058183473982592140598437877661775932667801749348425665604537
$+$ 0.059191993922961543783539007749154608870930698106730512140412
$+$ 0.059980315777503252090063987996517116173062890744995232889533
$+$ 0.060545506934737795138125251467754162753685843066200393415365
$+$ 0.060885464844856343881198614222696219728076992019447716770037
$+$ 0.060998924841205880159797643098356050557122808491557005025225

Gauss-Bond Abscissae for Order 50

$\pm$ 0.998909990848903495168995877273385637084249938575111612977051
$\pm$ 0.994261260436752574621084897949263128231784477968454277278541
$\pm$ 0.985915991735902996583885707558309632652115992556224484566752
$\pm$ 0.973903368019323867231755486394186639013106901689554435137002
$\pm$ 0.958267848613908194557707038316323727612444646454842233655520
$\pm$ 0.939067544002962383435367806390905144165164730768267936526322
$\pm$ 0.916373862309780230823571294251475879366043904795797069588505
$\pm$ 0.890271218029527303277795370736726893557801394449321043531130
$\pm$ 0.860856711182292371473495743716111323577346914638034911360632
$\pm$ 0.828239763823064832854818424016562364262390186679011920334688
$\pm$ 0.792541712099381205234410878375875730157325296801126404034282
$\pm$ 0.753895354485375525763960025452473987630640624374368957642043
$\pm$ 0.712444457577036644580524855400145643036948463177564539567964
$\pm$ 0.668343221175370086864460419403988875929612565078091525009347
$\pm$ 0.621755704600723273755042745403315950054391949288853476007318

$\pm$ 0.5728552163513038365223947025901909074209477599606780229435 83

$\pm$ 0.521823669366185842514087784826818024569355047379422926201926

$\pm$ 0.468850904286041063610457258811622468806727482427668738182181

$\pm$ 0.414133983226303877936871809744657907923981925685998693125257

$\pm$ 0.357876456688409509775201088519666356896388431823667655371603

$\pm$ 0.300287606335331939530245649644420333331342481297453907665457

$\pm$ 0.241581666447798703846733114869262423738796774456755816882615

$\pm$ 0.181977026957077545323998701169214449923922918920115452813381

$\pm$ 0.121695421018888766963820420963181108745305234967309681175283

$\pm$ 0.060961100150578724734194706843205404697031462796017398755700

Gauss-Bond Weights for order 50

+ 0.003406490112303860996809607645841513624438624407201118899429

+ 0.004370098755700379355538600044305012358167558810163335383832

+ 0.014353782018115907285202503974228885286249981393433619843981

+ 0.007239852121332318916128381001582557944008864516134082734013

+ 0.026749772601291364398962154797964462671028319948204192884085

+ 0.008669265725500378715810324048527088433034333848633654569400

+ 0.039861770729214394537033036722052063806006678543867601704276

+ 0.009029449190875200034597478278869459331326077290927533295858

+ 0.053174805415328024629925367193228765391930151479223367119109

+ 0.008591850070749660728044527197228946625716229570925195301750

+ 0.066242515512632951799320598077824643434659535980387744233169

+ 0.007590834418100947223279483016150722095696726267444352794058

+ 0.078658895749482658196352640558527369873961754795661650820781

+ 0.006239376589043140894183136195447632442868009255257059748002

+ 0.090053541132593820745671650626402390650406622869614994554503

+ 0.004731754066946247110288119772137674305139168545904700781417

+ 0.100094299719009232514063249890693400001589683152628530171250

+ 0.003241607602433693842224567422729145088937950096547898393271

+ 0.108492474242912522210201327003939285445470497177736073220374

+ 0.001918331288042398042039449242079002826556118981355641935613

$+$ 0.11500860019939111244420648747653046837824800595768153819619 3

$+$ 0.00088312300136854019843204749805618403273038835243089727672 4

$+$ 0.11945781939444517475801373044969554675658802688179421120282 3

$+$ 0.00022540080275078721365174066247462713820825553374873934041 0

$+$ 0.12171428954043528321001979120348315205703243634309226559567 9

3.9 Gauss-Bond Rules for Order 60

Gauss-Legendre Abscissae for order 60

$\pm$ 0.99921012322743602203422958579764926633894701281310018457913 5

$\pm$ 0.99584052511883817387674671337744065246015120843688430645383 6

$\pm$ 0.98978789522222171736727898701609604180662757170643620835369 7

$\pm$ 0.98106720175259818561857679982677003245691501297081130735410 1

$\pm$ 0.96970178876505273372154409891379427027412590025083911912885 7

$\pm$ 0.95572225583999610739723184582969949918855794266965203593881 8

$\pm$ 0.93916627611642324949541901160970509193287176375358078803481 1

$\pm$ 0.92007847617762755285665686251989682294600821175291955941390 0

$\pm$ 0.89851031081004594193778932957262832283636178034388167207218 3

$\pm$ 0.87451992264689831512930809991243579622224413212399376910095 0

$\pm$ 0.84817198478592963249051549499437559065775342188866366166229 8

$\pm$ 0.81953752616214575936851810851972355949176870756326092626774 5

$\pm$ 0.78869373993226405456994479977721572103929640806034083154424 4

$\pm$ 0.75572377530658568686884206660237308993022639941601851682546 6

$\pm$ 0.72071651335573039943602106101352106002685069439889646890266 6

$\pm$ 0.68376632738135543722293023922429713971337001404075619237878 6

$\pm$ 0.64497282848947706781344789642044469631391769127917284747190 5

$\pm$ 0.60444059704851036344420877631120169854286295326307575342305 1

$\pm$ 0.56227890075394453917827258748599870598566805149061309487032 9

$\pm$ 0.51860140005856974741788934848472127763636485651938285942276 0

$\pm$ 0.47352584176170711110816305375279461411933019148350993230564 5

± 0.427173741583078389307452853530311899752960182274512211069868
± 0.379670056576797977154952670521887681279636107907408513629701
± 0.331142848268448194252352965350552683420058782339075047412199
± 0.281722937423261691690694860339441578913734657621995079701599
± 0.231543551376029338010344631346755423912493826969835791508388
± 0.180739964873425417240876941261852617500318236583352668871860
± 0.129449135396945003146444164649575764589691098066910699931118
± 0.077809333949536569419285507082225281324964713253348714480806
± 0.025959772301247798589170385400344823550421052061839717226704

Gauss-Legendre Weights for order 60

+ 0.002026811968873758496431710209892324848840851984744707170176
+ 0.004712729926953568640894821714077236304069598458214092148953
+ 0.007389931163345455531516956022086061437572584508083690778470
+ 0.010047557182287984357885764377057257248941407645026178169276
+ 0.012678166476815960131495379269514235923957931732923673175077
+ 0.015274618596784799306726038098825336289870068570161705713211
+ 0.017829901014207720260396261248348568909556515974769827120197
+ 0.020337120729457286775032147417106302885929893667921010800370
+ 0.022789516943997819863783458192900209363500266416275843561915
+ 0.025180477621521248379570965972361280792053828512711121954932
+ 0.027503556749924791635223197638622249388998433602073827181600
+ 0.029752491500788945240836484673487712122157328952225883394393
+ 0.031921219019296328949458899536760478569143142598169866316142
+ 0.034003892724946422834914401555258742960656572178775836731821
+ 0.035994898051084503066578646288062340290932930595314574345759
+ 0.037888867569243444030940794209276033376603509717908973456510
+ 0.039680695452380799470122834811710010140109634921180530046437
+ 0.041365551235584755613163836806658898004493570749302614089038
+ 0.042938892835935641954231220656382807753502055274340978641498
+ 0.044396478795787113327784164091377370080301588931300831686766
+ 0.045734379716114486647196455290909332971179934852795069949067

+ 0.046948988848912204847013156394701565727359398008090813641724
+ 0.048037031819971180963666652728733652496919334448844660698695
+ 0.048995575455756835389475686857894296667064590009346006791514
+ 0.049822035690550181011159230893703310200413837363126270733343
+ 0.050514184532509374598238735741653640398735322723844810897149
+ 0.051070156069855627404549120734491032020665904755119158282529
+ 0.051488451500980933995044397177054307974567072248938936249973
+ 0.051767943174910187543803643028823728878595220111959828678770
+ 0.051907877631220639732864938362269675973307670486508677596032

Gauss-Bond Abscissae for Order 59

± 0.999210123227436022034229585797649266338947012813100184579135
± 0.995840525118838173876746713377440652460151208436884306453836
± 0.989787895222221717367278987016096041806627571706436208353697
± 0.981067201752598185618576799826770032456915012970811307354101
± 0.969701788765052733721544098913794270274125900250839119128856
± 0.955722255839996107397231845829699499188557942669652035938818
± 0.939166276116423249495419011609705091932871763753580788034811
± 0.920078476177627552856656862519896822946008211752919559413900
± 0.898510310810045941937789329572628322836361780343881672072183
± 0.874519922646898315129308099912435796222244132123993769100950
± 0.848171984785929632490515494994375590657753421888663661662298
± 0.819537526162145759368518108519723559491768707563260926267745
± 0.788693739932264054569944799777215721039296408060340831544244
± 0.755723775306585686868842066602373089930226399416018516825466
± 0.720716513355730399436021061013521060026850694398896468902666
± 0.683766327381355437222930239224297139713370014040756192378786
± 0.644972828489477067813447896420444696313917691279172847471905
± 0.604440597048510363444208776311201698542862953263075753423051
± 0.562278900753944539178272587485998705985668051490613094870329
± 0.518601400058569747417889348484721277636364856519382859422760
± 0.473525841761707111108163053752794614119330191483509932305645

$\pm$ 0.427173741583078389307452853530311899752960182274512211069868
$\pm$ 0.379670056576797977154952670521887681279636107907408513629701
$\pm$ 0.331142848268448194252352965350552683420058782339075047412199
$\pm$ 0.281722937423261691690694860339441578913734657621995079701599
$\pm$ 0.231543551376029338010344631346755423912493826969835791508388
$\pm$ 0.180739964873425417240876941261852617500318236583352668871860
$\pm$ 0.129449135396945003146444164649575764589691098066910699931118
$\pm$ 0.077809333949536569419285507082225281324964713253348714480806
0.000000000000000000000000000000000000000000000000000000000000

Gauss-Bond Weights for order 59

$+$ 0.002037405059781092882775018474023264391992315542419244531516
$+$ 0.004675568485308896042403253020893055630241068597634110862091
$+$ 0.007463180379811835757384497087616362806637857175866611700538
$+$ 0.009930485355364355830115370807077500392835916365064464650535
$+$ 0.012845993798750910430009828101207048341291203563275966025692
$+$ 0.015049472862989841058607633429545626135615321426681748365563
$+$ 0.018118819682611922323759394615738222848656599785977871187398
$+$ 0.019977888633966454633613473398260792901770187152229842560096
$+$ 0.023225857758441909614509883031314719362881467618365197208598
$+$ 0.024659812369184982622074982202734986495719413684492761736883
$+$ 0.028116360701644016171246370388098310356773917631488071204936
$+$ 0.029038931184933122479972209934450483718667650223711733462141
$+$ 0.032745204231132649844361583633977593861280525063289260546410
$+$ 0.033058453035661654925898798736785968224094298960871447745217
$+$ 0.037074583367446276737260394147813729736870888818994896260164
$+$ 0.036659853171451189234840211085900106827082465135628113036260
$+$ 0.041077135454523524580079899015970548188138837007391360317474
$+$ 0.039779572494528297999797802682725383840380555471591101970234
$+$ 0.044741975177604680971313248837412648662719896907452094515798
$+$ 0.042341161913023765196176792224274386436925320488607913807302
$+$ 0.048087851639046183375115924146512139053797465738624073428488

$+$ 0.044235535514689912954294739211894077884024418783027752133851
$+$ 0.051196722937056266632474665196848150700597430522270991211255
$+$ 0.045263881900390932120552445362493860289495190413580523552437
$+$ 0.054319787690649039772049522968811688925198935261083147636355
$+$ 0.044927265185273887971151965150582195602693534062972091705511
$+$ 0.058345965533273020595165354157716051305439649971218651850631
$+$ 0.041204735430283251178634686828924887178152670527549834188583
$+$ 0.0690161515558670384253834210451784193063366833847521407157053
$+$ 0.081568774985011483621051683337224033673316328502235430881978

3.10 Gauss-Bond Rules for Order 61

Gauss-Legendre Abscissae for order 61

$\pm$ 0.999235597631363471731862259569130862990434594753219222878506
$\pm$ 0.995974599815120234268012760712288277006491299276658013646896
$\pm$ 0.990116745232517050965531685699916503387787494959196021909323
$\pm$ 0.981676011284037079685172540751319425400238049049369259208522
$\pm$ 0.970674258833182908247408453547910277709258784394341822584402
$\pm$ 0.957140151912984091372080287421215268536529702908706333732885
$\pm$ 0.941108986681361147477543860977789018357664422718456180858112
$\pm$ 0.922622581382955261257551500555845458482943663704988712436695
$\pm$ 0.901729162474001170642040042683511180899045862239966348504776
$\pm$ 0.878483237214881032478947513893352651118376033764342334780944
$\pm$ 0.852945450847663445564846904674170561543993163060401386038751
$\pm$ 0.825182428108659950664281892450847553813448490159997955990247
$\pm$ 0.795266599282359649152048802752546295259334932118603839266708
$\pm$ 0.763276011172312197145914668494350404897306959097495050402808
$\pm$ 0.729294123449465109688956690135531835689438764265070914897764
$\pm$ 0.693409590894491155499184363960231777352929074589478586778370
$\pm$ 0.655716032095070871699185777836306929619144102213489896986644

$\pm$ 0.616311785197921724709616859328591080556715210924893833257767

$\pm$ 0.575299651350830618600369981944972925095255195135103357677289

$\pm$ 0.532786626502925265638481716373071687160449497827962707374314

$\pm$ 0.4888836222622252118820698511425829644306564086282179498146411

$\pm$ 0.443705176538531601995589301065822605272548703395989944367172

$\pm$ 0.397369154725756609178291847368589265895836518900991990339747

$\pm$ 0.349996442204066834533434477062093891125003107937724667275988

$\pm$ 0.301710628963030712604486525456977162891789185456347863725588

$\pm$ 0.252637687169053495833690863334405139278934657671083299350784

$\pm$ 0.202905642518058499226947203343057082107164996910890410477583

$\pm$ 0.152644240230815300529506761773477438301725345063889743629893

$\pm$ 0.101984606562274068957208404764365656608815983437367639161095

$\pm$ 0.051058906707974349366887500618900805494974938824405711524320

0.0000000000000000000000000000000000000000000000000000000000000

Gauss-Legendre Weights for order 61

+ 0.001961453361670282671773064461926847742828817336308117069429

+ 0.004560924006012417184538932579742787457913388017555594164382

+ 0.007152354991749089585833993091694174307911270832319955258854

+ 0.009725461830356133736136671755570116631624047104659094048291

+ 0.012273263507812104629277800359526129253061699163132152814873

+ 0.014789065884937914546178515781488258785544672271145106616312

+ 0.017266292987613743594433212772436618376130575992849183538161

+ 0.019698477746101181330517830212684563572200865044543627142239

+ 0.022079273148319044002475765628181624112399213342092774535878

+ 0.024402467187544202915340659732467528834916958237834797911194

+ 0.026661998524150889662810706844968015976899109048739846958612

+ 0.028851972088183401504341775545590248938273228676866280067340

+ 0.030966674368397394824697946612489380228601174980792613680228

+ 0.033000588275907410632723683441004489427835644537962895432978

+ 0.034948407516533351090851919209219275130450131764505955880369

+ 0.036805050423154817384321042864067458106353558235042860918546

$+$ 0.038565673207008172746152047629653800516201437695204194900123
$+$ 0.040225682590998247367639983757511577399987464745196362868622
$+$ 0.041780747790888492066675572351073006410490307909681123011769
$+$ 0.043226811812496097901043645822682872991180133222891147925070
$+$ 0.044560102035083488271541419831088353395558401483019090610803
$+$ 0.045777140053145959371339833846824793307904588784925193624984
$+$ 0.046874750750809065976429445588590738207075564466903811008577
$+$ 0.047850070585095607161833427518699665931592965803301924432322
$+$ 0.048700555056411526087530088314705610909528191060430627521697
$+$ 0.049423985346735589939968776651116504453252996642389804474125
$+$ 0.050018474108178253425051615006358631279405726054952198831311
$+$ 0.050482470386797404648144465188151275686689204094492265811337
$+$ 0.050814763668818343207700529223478704376641350069361591149420
$+$ 0.051014487038697263543735080573852018119751623619109909469795
$+$ 0.051081119440786217977921095606309860263591379523579796644674

Gauss-Bond Abscissae for Order 60

$\pm$ 0.999235597631363471731862259569130862990434594753219222878506
$\pm$ 0.995974599815120234268012760712288277006491299276658013646896
$\pm$ 0.990116745232517050965531685699916503387787494959196021909323
$\pm$ 0.981676011284037079685172540751319425400238049049369259208522
$\pm$ 0.970674258833182908247408453547910277709258784394341822584402
$\pm$ 0.957140151912984091372080287421215268536529702908706333732885
$\pm$ 0.941108986681361147477543860977789018357664422718456180858112
$\pm$ 0.922622581382955261257551500555845458482943663704988712436695
$\pm$ 0.901729162474001170642040042683511180899045862239966348504776
$\pm$ 0.878483237214881032478947513893352651118376033764342334780944
$\pm$ 0.852945450847663445564846904674170561543993163060401386038751
$\pm$ 0.825182428108659950664281892450847553813448490159997955990247
$\pm$ 0.795266599282359649152048802752546295259334932118603839266708
$\pm$ 0.763276011172312197145914668494350404897306959097495050402808
$\pm$ 0.729294123449465109688956690135531835689438764265070914897764

$\pm$ 0.693409590894491155499184363960231777352929074589478586778370
$\pm$ 0.655716032095070871699185777836306929619144102213489896986644
$\pm$ 0.616311785197921724709616859328591080556715210924893833257767
$\pm$ 0.575299651350830618600369981944972925095255195135103357677289
$\pm$ 0.532786626502925265638481716373071687160449497827962707374314
$\pm$ 0.488883622262252118820698511425829644306564086282179498146411
$\pm$ 0.443705176538531601995589301065822605272548703395989944367172
$\pm$ 0.397369154725756609178291847368589265895836518900991990339747
$\pm$ 0.349996442204066834533434477062093891125003107937724667275988
$\pm$ 0.301710628963030712604486525456977162891789185456347863725588
$\pm$ 0.252637687169053495833690863334405139278934657671083299350784
$\pm$ 0.202905642518058499226947203343057082107164996910890410477583
$\pm$ 0.152644240230815300529506761773477438301725345063889743629893
$\pm$ 0.101984606562274068957208404764365656608815983437367639161095
$\pm$ 0.051058906707974349366887500618900805494974938824405711524319

Gauss-Bond Weights for order 60

$+$ 0.001570151245745902517777981670094260889512314871221847093653
$+$ 0.005929089576786505240727589423047401596927958164966660393316
$+$ 0.004471680966508510720811723689350519126034891639581914022637
$+$ 0.013972749275551051072697164799547022572341560082962146465776
$+$ 0.006254016675066981729856477121578249936681508238236012877501
$+$ 0.022749501830355553619071368209427930102562056496574178192 72
$+$ 0.007225240671561520868393516477165572763006623020473916706528
$+$ 0.031933384792810778893279732766414026256700264870949789090010
$+$ 0.007561190944500750276825907748679520819462885112273183487246
$+$ 0.041270670600975908260916580171148799693251479795880818908759
$+$ 0.007397970459739966584045657958716422631384492729094041551630
$+$ 0.050537182287831469517872227929436565225176764605666819816982
$+$ 0.006854503508503150767429952314607159965156046430804615488607
$+$ 0.059526622683816032024476928309490730807035784627901442698889
$+$ 0.006039799968228011967730829015535346622882632728518997563930

$$
\begin{aligned}
&+ \quad 0.068047436990277278320129078575812580160046156771770401025039 \\
&+ \quad 0.005055089834037838479833821657753556795890130639798061380344 \\
&+ \quad 0.075922846551242593631895697904066011727987717164538993774692 \\
&+ \quad 0.003993852977492110478475965335846360499769694198356091065308 \\
&+ \quad 0.082992205812111072436497293461815435396178606602076088477607 \\
&+ \quad 0.002940917517081863962982707854295420928995735502218074964722 \\
&+ \quad 0.089112888091911825897868241074640038779197605660286352690490 \\
&+ \quad 0.001971173183389043133954204727471753709094242526435670828432 \\
&+ \quad 0.094162298447890651572975302289901989573808378361827844731106 \\
&+ \quad 0.001148192396813666680816083597558368604374991094903435531290 \\
&+ \quad 0.098039783266023226472438422208457176977053819365164571775219 \\
&+ \quad 0.000522933635653144242271895203845860746023475906136770830331 \\
&+ \quad 0.100668291919413484858993032533157098609003559963517592967995 \\
&+ \quad 0.000132639009558585504294889298190461950402952879624256763981 \\
&+ \quad 0.101995694879121518521823958061433493626361520795156169208901
\end{aligned}
$$

4 Conclusion

A new integration method has been proposed which provides maximum accuracy for the number of function calls along with a means to estimate the error. The error estimation metric should be carefully constructed to make the most of the difference between the integrals, but no optimal estimation formula is known. One approach might be to simply compare the number of significant digits in agreement between the two estimates. If more that three quarters of the digits agree, the Gauss-Legendre estimate is likely to be very good. The only downside of this approach is that even if the number of matching digits is low, the Gauss-Legendre estimate may still be very good.

If the error estimate meets the users requirement, the value of the integral can be accepted with the knowledge that it has been found with minimal function calls. If not, simply revert to a Gauss-Patterson scheme starting from the already computed Gauss-Legendre integral. Note that there is no function call penalty imposed by with this strategy.

The above tables provide the user with a tool for obtaining error estimates from a single Gauss-Legendre integration without requiring additional function calls. Hence, it is now possible to estimate the accuracy of a single, non-adaptive, integration. Based on that estimate, the result of the integration can be accepted or a progressive method initiated with minimal overhead.