Lensmaker's Formula

by C. Bond

Lenses with the same shape and index of refraction will have the same focal length. the *lensmaker's formula* relates the index of refraction, the radii of curvature of the two surfaces of the lens, and the focal length of the lens.

A number of idealizations, simplifications and approximations are used to complete the derivation, but the results are compact and sufficiently accurate for most purposes.

We begin by observing that a lens with convex surfaces behaves the same as two plano-convex lenses placed with the flat sides in contact. Fig. (1) shows the division of the lens into two pieces which we will analyze separately.

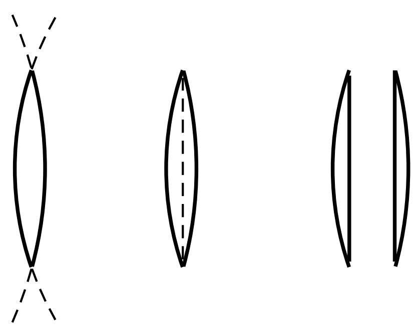

Figure 1: Separation of Lens into Halves

Recall that with thin lenses we can reverse the direction of the ray without affecting the incident and refracted angles. Hence, Fig. (2) which represents one plano-convex lens may be regarded as the rightmost half of the original lens or the leftmost half reversed. In this figure, a perpendicular ray enters the flat surface of the lens. It proceeds to the curved surface without initial refraction. When it emerges from the curved surface it is refracted by an angle determined by Snell's law. The radius from the center of curvature extended through the exit point determines the surface normal. The angle in the media between the ray and the normal is θ_1. The angle between the refracted ray and the normal is θ_2.

If the index of refraction of the lens is n and we take the index of refraction

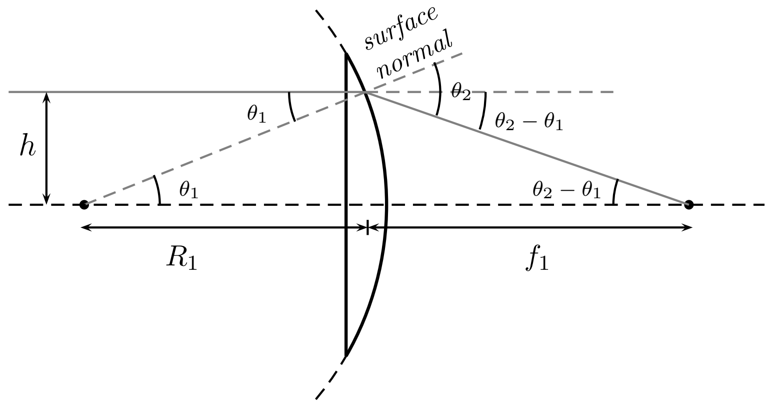

Figure 2: Ray Diagram for Lens Analysis

of air as 1, Snell's law holds that

$$n \sin\theta_1 = \sin\theta_2.$$

Assuming small angles (paraxial rays), we now approximate the sines of the angles with the angles themselves so that

$$n\,\theta_1 \approx \theta_2.$$

Substituting this in the angle between the refracted ray and the axis

$$\theta_2 - \theta_1 = n\,\theta_1 - \theta_1 = (n-1)\theta_1. \tag{1}$$

For these small angles, the tangents are also close to the angles themselves. We can write

$$\theta_2 - \theta_1 \approx \frac{h}{f_1}, \tag{2}$$

and

$$\theta_1 \approx \frac{h}{R_1}. \tag{3}$$

Eliminating h between Eq. (2) and Eq. (3) and substituting from (1),

$$\frac{1}{f_1} = \frac{n-1}{R_1}. \tag{4}$$

Substituting from the lens equation[1] which relates the object and image distances to the focal length

$$\frac{1}{o_1} + \frac{1}{i_1} = \frac{n-1}{R_1}. \tag{5}$$

An equivalent analysis of the other half of the lens gives

$$\frac{1}{o_2} + \frac{1}{i_2} = \frac{n-1}{R_2}. \tag{6}$$

We can now combine (5) and (6) noting that the image of the first lens is a virtual object for the second lens. Therefore $i_1 = -o_2$ and, adding the two equations,

$$\frac{1}{o_1} + \frac{1}{i_2} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{7}$$

Writing the lens equation in terms of the object and image distances,

$$\frac{1}{o} + \frac{1}{i} = \frac{1}{f}. \tag{8}$$

But o_1 and i_2 are the object and image distances of the whole lens, so $o_1 = o$ and $i_2 = i$. Thus,

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{9}$$

which is the **lensmaker's formula**.

Considering the approximations used, we should not expect this formula to be accurate for large angles of incidence, but for many purposes it is quite useful.

[1]See paper on *Derivation of Focal Relations.*