Notes on “L” (Optimal) Filters

by C. Bond, 2011

1 Background

In 1959 A. Papoulis[1] published a paper which completed the description of a new class of filters with ‘optimal’ properties, originally reported in 1958. In particular, these low-pass filters exhibited the maximum possible rolloff consistent with monotonic magnitude response in the pass band. The problem under consideration was that Butterworth filters had monotonically decreasing amplitude in the pass band, but did not exhibit the aggressive rolloff associated with Chebyshev filters. On the other hand, Chebyshev filters exhibited amplitude ripples in the pass band, which in some applications was undesirable.

Papoulis found that the Legendre polynomials of the 1st kind[2], can form the basis for a suitable set of transfer functions. The new polynomials, $L_n(\omega^2)$, are related to, but not the same as, the Legendre polynomials from which they are derived.

In the following paragraphs, the generation of these polynomials will be discussed and examples of their application to the filter design problem will be given.

2 Legendre Polynomials

Legendre polynomials, usually notated as $P_n(x)$, belong to the class of orthogonal polynomials which include the Chebyshev and Laguerre Polynomials.

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[1] A. Papoulis, *On Monotonic Response Filters*, Proc. IRE, 47, Feb. 1959, 332-333

[2] Abramowitz and Stegun, *Handbook of Mathematical Functions*, Dover, 1970, (332-344,798)

The first few are

$$
\begin{aligned}
P_0(x) &= 1 & (1)\\
P_1(x) &= x & (2)\\
P_2(x) &= \frac{1}{2}(3x^2-1) & (3)\\
P_3(x) &= \frac{1}{2}(5x^3-3x) & (4)\\
P_4(x) &= \frac{1}{8}(35x^4-30x^2+3) & (5)\\
&\cdots
\end{aligned}
$$

Formulas and recurrence relations for the Legendre polynomials are given in *Abramowitz and Stegun.*

3 Optimal Polynomials

The L_n polynomials have the following properties:

$$
\begin{aligned}
L_n(0) &= 0\\
L_n(1) &= 1\\
\frac{d\,L_n(\omega^2)}{d\omega} &\leq 0 \qquad \text{(monotonic decreasing)}\\
\left.\frac{d\,L_n(\omega^2)}{d\omega}\right|_{\omega=1} &= M \qquad \text{(M maximum)}
\end{aligned}
$$

Papoulis' original paper described only the odd ordered L_n polynomials. In a later paper he completed the exposition to include the even ordered polynomials.

For the odd order he gave

$$L_n(\omega^2) = \int_{-1}^{2\omega^2-1} \left[\sum_{i=0}^{k} a_i P_i(x)\right]^2 dx \tag{6}$$

where $n = 2k + 1$ and the $P_i(x)$ are the Legendre polynomials of the first kind with the constants a_i given by $(i = 0, 1, 2, \cdots, k)$

$$a_0 = \frac{a_1}{3} = \frac{a_2}{5} = \cdots = \frac{a_i}{2i+1} = \frac{1}{\sqrt{2}(k+1)}. \tag{7}$$

For example, $L_3(\omega^2)$ has order $n = 3$. Thus $k = (n-1)/2 = 1$. The index, i, for this polynomial takes on consecutive values 0 and 1. We have,

$$a_0 = \frac{1}{\sqrt{2}(k+1)} = \frac{1}{\sqrt{2}(2)}, \tag{8}$$

and

$$a_1 = \frac{3}{\sqrt{2}(2)}. \tag{9}$$

For even ordered L_n Papoulis found

$$L_n(\omega^2) = \int_{-1}^{2\omega^2-1} (x+1) \left[\sum_{i=0}^{k} a_i P_i(x) \right]^2 dx \tag{10}$$

where $n = 2k + 2$.

For the a_i there are two cases to consider:

Case 1 (k even, $i = 0, 2, 4, \cdots, k$)

$$a_0 = \frac{a_2}{5} = \frac{a_4}{9} = \cdots = \frac{a_i}{2i+1} = \frac{1}{\sqrt{(k+1)(k+2)}}$$
$$a_1 = a_3 = a_5 = \cdots = a_{i-1} = 0$$

Case 2 (k odd, $i = 1, 3, 5, \cdots, k$)

$$\frac{a_1}{3} = \frac{a_3}{7} = \frac{a_5}{11} = \cdots = \frac{a_i}{2i+1} = \frac{1}{\sqrt{(k+1)(k+2)}}$$
$$a_0 = a_2 = a_4 = \cdots = a_i = 0$$

4 Filter Design

The design and analysis of filters is a branch of the broader subject of network theory, which is concerned with the behavior of certain functions in the complex s plane, where $s = \sigma + j\omega$. The specific function values of interest include the magnitude along the imaginary axis. For this restricted case, $s = j\omega$, and $\sigma = 0$.

The magnitude response of any low-pass, all pole filter can be given by

$$M(\omega) = \frac{K_0}{\sqrt{1 + f(\omega^2)}}. \tag{11}$$

The L_n polynomials form the desired filters from the rational function

$$M^2(\omega) = \frac{1}{1 + L_n(\omega^2)} \tag{12}$$

where n is the order of the filter. This is a magnitude squared function.

The first step in transforming the magnitude squared function to the s plane uses the following relations:

$$\begin{aligned} s &= j\omega \\ s^2 &= -\omega^2 \end{aligned}$$

or, what is the same thing $\omega^2 = -s^2$. Substituting in (12) we have

$$h(s^2) = H(s)H(-s) = \frac{1}{1 + L_n(-s^2)}. \tag{13}$$

In network theory, the factoring schemata $H(s)H(-s)$ is used to isolate the positive real factors from their negative real counterparts. The reason has to do with physical realizability. In order for the transfer function to represent a stable network, the poles must be in the left half plane or on the $j\omega$ axis. Moreover, poles on the $j\omega$ axis must be simple.

$H(s)$ has all poles in the left half plane or on the $j\omega$ axis. Its denominator is the so-called Hurwitz polynomial which has zeros at the pole locations of $H(s)$.

To find $H(s)$ simply find all roots to the denominator polynomial of $h(s^2)$ and discard those roots which represent poles in the right half plane.

5 Example

We will find the transfer function for the third order Optimal filter.

The third order Optimal polynomial is

$$L_3(\omega^2) = 3\omega^6 - 3\omega^4 + \omega^2, \tag{14}$$

so the corresponding filter magnitude squared function is

$$M^2(\omega) = \frac{1}{1 + \omega^2 - 3\omega^4 + 3\omega^6}. \tag{15}$$

Substituting $-s^2$ for ω^2 gives

$$h(s^2) = H(s)H(-s) = \frac{1}{1 - s^2 - 3s^4 - 3s^6}. \tag{16}$$

The denominator polynomial can be factored into the form

$$1 - s^2 - 3s^4 - 3s^6 = (s + r_1)(s + r_2)(s + r_3)(s + r_4)(s + r_5)(s + r_6) \tag{17}$$

once the roots, r_n, have been found. Now the r_n are the zeros of the denominator polynomial and, hence, poles of the transfer function. We want none of these to occur in the right half plane, so we discard those values with negative real parts. (If r_1 is negative, s would have to be positive to form a zero.)

We find that the Hurwitz polynomial is constructed from the following roots

$$\begin{aligned} s &= 0.34518561903119696 - j0.90086563551837810, \\ s &= 0.34518561903119696 + j0.90086563551837810, \\ s &= 0.62033181713012371, \end{aligned}$$

where the complex roots appear in conjugate pairs, as expected.

Now form the Hurwitz polynomial and complete the transfer function.

$$H(s) = \frac{0.5773502691896}{0.5773502691896 + 1.3589712494455s + 1.3107030551925s^2 + s^3}, \tag{18}$$

where the value in the numerator is chosen for unity response at DC and the number of significant digits in each coefficient is unnecessarily large.

From this point, the locations of the poles are known and parameters of interest can be found using standard methods for any low-pass filter.

Appendix A

Here is a list of the first few "L" polynomials.

n	$L_n(\omega^2)$
1	ω^2
2	ω^4
3	$\omega^2 - 3\,\omega^4 + 3\,\omega^6$
4	$3\,\omega^4 - 8\,\omega^6 + 6\,\omega^8$
5	$\omega^2 - 8\,\omega^8 + 28\,\omega^6 - 40\,\omega^8 + 20\,\omega^{10}$
6	$6\,\omega^4 - 40\,\omega^6 + 105\,\omega^8 - 120\,\omega^{10} + 50\,\omega^{12}$
7	$\omega^2 - 15\,\omega^4 + 105\,\omega^6 - 355\,\omega^8 + 615\,\omega^{10} - 525\,\omega^{12} + 175\,\omega^{14}$
8	$10\,\omega^4 - 120\,\omega^6 + 615\,\omega^8 - 1624\,\omega^{10} + 2310\,\omega^{12} - 1680\,\omega^{14} + 490\,\omega^{16}$
9	$\omega^2 - 24\,\omega^4 + 276\,\omega^6 - 1624\,\omega^8 + 5376\,\omega^{10} - 10416\,\omega^{12} + 11704\,\omega^{14} - 7056\,\omega^{16} + 1764\,\omega^{18}$
10	$15\,\omega^4 - 280\,\omega^6 + 2310\,\omega^8 - 10416\,\omega^{10} + 27860\,\omega^{12} - 45360\,\omega^{14} + 44100\,\omega^{16} - 23520\,\omega^{18} + 5292\,\omega^{20}$