Problems and Solutions in Elementary Physics

by C. Bond

The following sections include solutions to a number of my favorite problems in elementary physics. Some of the solutions bear aspects resembling that of a magician pulling a rabbit out of a hat. Others simply demonstrate the remarkable power of a few seminal concepts to reveal the inner workings of the real world.

Most of the problems yield to solution strategies other than the ones shown, but these represent my own preference.

At some point in time, I expect to post similar documents containing problems of a more advanced nature, but the problems here may interest physicists and students at all levels.

Contents

1 Kinematics Equations

Kinematics deals with problems involving distance, velocity, time and constant acceleration. The restriction that acceleration is a constant for these problems limits the scope of this subject, but a large body of applications remains. Vector concepts are not generally employed, so that velocity is undirected and equivalent to speed. Distance, denoted by x, refers to the total distance travelled, not necessarily the distance between the starting and stopping points. *Force* and *mass* are not involved in the kinematics relations.

The first equation relates the distance covered by an object during some time interval. Since the acceleration may be non-zero, the velocity may vary during the time interval under consideration. The most useful relation is

$$\overline{v} = \frac{x}{t}, \tag{1.1}$$

where $\overline{v}$ is the *average velocity*, x is the *total distance* and t is the *elapsed time*.

Given that acceleration is to be constant, velocity may be uniformly increasing or decreasing. A plot showing the case of increasing velocity is shown in Fig. (1.1).

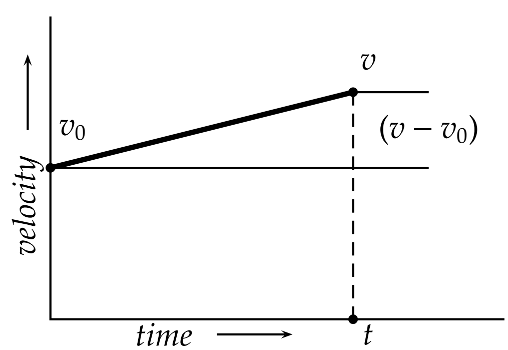

Figure 1.1: Velocity Under Constant Acceleration

The relation between acceleration and velocity is

$$\begin{aligned} a &= \frac{v - v_0}{t}, \qquad \text{or} \\ v &= v_0 + at, \end{aligned} \tag{1.2}$$

where v is the *final velocity* after the specified time has elapsed, v_0 is the *initial velocity* and a is the (constant) *acceleration*.

The *average velocity* for this case is

$$\overline{v} = v_0 + \frac{v - v_0}{2} = \frac{v + v_0}{2} \tag{1.3}$$

Other useful equations can be derived from these elementary relations. It is customary to develop a set of equations which involve only three of the four quantities *distance, velocity, acceleration* and *time*. We already have an equation relating *velocity, acceleration* and *time*, Eq. (1.2).

An equation involving *distance, velocity* and *time* requires substituting Eq. (1.3) for $\overline{v}$ in Eq. (1.1).

$$x = \frac{v + v_0}{2} t$$

We may now substitute Eq. (1.2) for v in Eq. (1) to derive an equation relating *distance, acceleration* and *time*.

$$\begin{aligned} x &= \frac{v_0 + at + v_0}{2} t \\ x &= v_0 t + \frac{1}{2} a t^2 \end{aligned} \tag{1.4}$$

Eq. (1.2) can be rearranged to isolate t and then substituted for t in Eq. (1) for an equation relating *distance, velocity* and *acceleration*.

$$\begin{aligned} x &= \frac{v + v_0}{2} \frac{v - v_0}{a} \\ x &= \frac{v^2 - v_0^2}{2a} \end{aligned} \tag{1.5}$$

A more convenient form for this equation is

$$v = \sqrt{v_0^2 + 2ax}, \tag{1.6}$$

where v_0 is often zero.

1.1 Miscellaneous Problems in Kinematics

For some of the following problems the constant acceleration is due to gravity and will be notated as $a = g = 32\ \text{ft/sec}^2$. '$g$' may be positive or negative depending on the context. Another quantity introduced is the coefficient of static friction, μ, which represents the relative amount of normal force which must be overcome in horizontal motion and typically varies from 0 to 1. When $\mu = 1$ it takes as much force to slide the object as it does to lift it.

1.1.1 Minimum Time for a Vehicle to go from 0 to 60 mph.

The relevant equation is $v = at$, where $a = \mu g$. It is common practice to assume that the maximum practical value of the coefficient of friction, μ, for rubber tires on pavement is unity. In this case, converting *mph* to *fps*, we have 60 mph = 88 fps, so

$$88 = 32\,t$$

and $t = 2.75$ sec.

1.1.2 Minimum Stopping Distance

Suppose we want to determine the minimum stopping distance of an automobile traveling at 60 mph. We again assume that the maximum value of μ is unity. Then the maximum deceleration is $-g$. We find, from Eq. (1.5),

$$x = \frac{v^2}{2g} = \frac{7744}{64} = 121\ \text{ft.}$$

Note that problems of uniform deceleration and acceleration differ by the negative sign. We could have found the stopping time as the same as for the previous problem and found the distance from $x = \overline{v}\,t = 44 \times 2.75 = 121$ ft.

1.1.3 Flight of the Bumblebee

Railroad train T_a leaves station A, at a uniform speed of 30 mph toward station B. Train T_b leaves station B at a uniform speed of 20 mph toward station A. The stations are 50 miles apart.

When T_a starts, a bumblebee which had been resting on its front begins flying toward T_b at 60 mph. When the bee hits T_b it reverses direction and heads back to T_a. It continues these alternations until the trains collide.

How far does the bee travel?

This problem is simple but instructive, because it invites the unwary to try a variety of unnecessarily complicated solution techniques. The essential point is that the trains travel 50 miles and with the speeds given, the trip will take 1 hour. But the bee travels at 60 mph, so the bee travels 60 miles.

2 Bouncing Ball

This interesting problem is not likely to be posed in your favorite physics text, but it illustrates the value of mathematical concepts in physics.

A certain rubber ball has been found to exhibit a *coefficient of restitution*, $c = 0.9$. This coefficient is the ratio of an objects velocity just after and just before a collision (bounce). Then $c = v_1/v_0$, where v_0 is the velocity before the bounce and v_1 is the rebound velocity.

From the kinematics equation, $v = \sqrt{2gh}$, for motion under the influence of gravity, we find

$$c = \sqrt{\frac{h_1}{h_0}}.$$

The ball will be dropped on a hard surface and the following problems will be solved: 1) What total distance will the ball travel before it stops? and 2) What is the total time the ball is in motion?

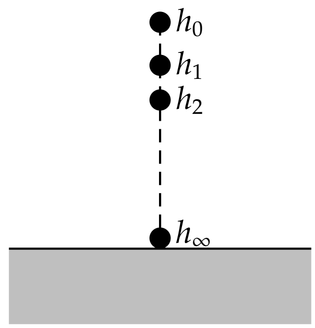

Figure 2.1: Bouncing Ball on Hard Surface

Let the initial height from which the ball is dropped be h_0. Then the peak height on the first bounce is $h_1 = c^2 h_0$. Similarly, the peak height on the second bounce is $h_2 = c^2 h_1 = c^4 h_0$. The total distance covered by the bouncing ball is then

$$d = h_0 + 2c^2 h_0 + 2c^4 h_0 + 2c^6 h_0 + \cdots \quad (2.1)$$

$$d = h_0 + 2c^2 h_0 (1 + c^2 + c^4 + \cdots) \quad (2.2)$$

Let

$$S = 1 + c^2 + c^4 + \cdots$$

then,

$$\begin{aligned} S &= 1 + c^2(1 + c^2 + c^4 + \cdots) \\ S &= 1 + c^2 S \qquad \text{so,} \\ S - c^2 S &= 1 \\ S(1 - c^2) &= 1 \qquad \text{and} \quad \text{finally,} \\ S &= \frac{1}{1 - c^2} \end{aligned} \quad (2.3)$$

We now have,

$$\begin{aligned} d &= h_0 + 2c^2 h_0 S \\ d &= h_0 + \frac{2c^2 h_0}{1 - c^2}. \end{aligned} \quad (2.4)$$

Given the initial height, h_0, the distance can now be found. For example, if $h_0 = 6$ft, $d = 57.14$ ft.

We now determined the elapsed time. From the kinematics equation, $v = g\,t$, for an object moving under the influence of gravity we have,

$$c = \frac{v_1}{v_0} = \frac{t_1}{t_0},$$

where t_0 is the time to fall from the initial height to the surface and t_1 is the time to reach the peak of the first bounce. So $t_1 = c\,t_0$.

Then

$$\begin{aligned} t &= t_0 + 2t_1 + 2t_2 + 2t_3 + \cdots \\ t &= t_0 + 2c\,t_0 + 2c^2\,t_0 + 2c^3\,t_0 + \cdots \\ t &= t_0 + 2t_0(c + c^2 + c^3 + \cdots) \end{aligned} \tag{2.5}$$

Now let

$$\begin{aligned} S &= c + c^2 + c^3 + \cdots \\ S &= c + c(c + c^2 + c^3 + \cdots) \qquad \text{then,} \\ S &= c + c\,S \qquad \text{so,} \\ S - c\,S &= c \qquad \text{and} \\ S &= \frac{c}{1-c}. \end{aligned} \tag{2.6}$$

We have

$$t = t_0 + 2t_0\,S = t_0 + \frac{2c\,t_0}{1-c}.$$

From the kinematics equation, $h = \frac{1}{2}g\,t^2$,

$$t_0 = \sqrt{\frac{2h_0}{g}}.$$

Finally,

$$t = t_0\left(1 + \frac{2c}{1-c}\right) = \sqrt{\frac{2h_0}{g}}\left(1 + \frac{2c}{1-c}\right) \tag{2.7}$$

From the previous problem with initial height, $h_0 = 6$ft,

$$t = 11.64\,\text{sec}.$$

3 Maximum Velocity in a Quarter Mile

To determine the maximum speed possible for a wheel driven vehicle, we will assume that the coefficient of friction between the tires and the ground is unity. For this case, the maximum acceleration is one g. We can use a simple equation from kinematics to solve for v_{max},

$$v_{\text{max}} = \sqrt{2gd},$$

where d is the distance. Then

$$\begin{aligned} v_{\text{max}} &= \sqrt{2 \times 32.2 \times 1320} \\ v_{\text{max}} &= 291.6\,\text{ft/sec} = 198.8\,\text{mi/hr} \qquad (3.1) \end{aligned}$$

It was long held that the assumption of unity coefficient of friction was appropriate for a wheel driven vehicle with rubber tires. However, this is incorrect if the tires develop significant viscous friction against the road surface. In fact, the viscous friction developed by melting rubber has a coefficient proportional to velocity — the faster the tires rotate, the greater the motive force.

With the development of dragster engines capable of spinning the wheels at high rates the maximum speed limit calculated above has been completely shattered. The current record is greater than 300 mph with no end in sight!

4 Rolling Up A Ramp

Here are a few problems which involve rotational kinetic energy.

4.1 Maximum Height of Ball

A solid ball is rolled toward a ramp. How high will it be when it stops and begins to roll back down?

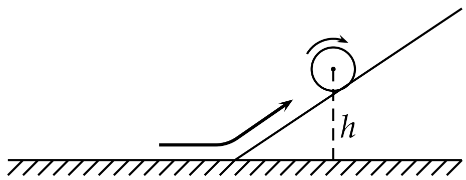

Figure 4.1: Ball Rolling Up A Ramp

The linear kinetic energy as the ball approaches the ramp is

$$K.E._{linear} = \frac{1}{2}mv^2. \tag{4.1}$$

Since the ball rolls without slipping, $v = \omega R$.

The rotational kinetic energy is

$$\begin{aligned} K.E._{rotational} &= \frac{1}{2}\left(\frac{2}{5}mR^2\right)\omega^2 \\ &= \frac{1}{5}mv^2, \end{aligned} \tag{4.2}$$

so the potential energy at the top of the rise is

$$\begin{aligned} P.E. &= K.E._{linear} + K.E._{rotational} \\ mgh &= \frac{1}{2}mv^2 + \frac{1}{5}mv^2 \\ mgh &= \frac{7}{10}mv^2 \\ h &= \frac{7}{10}\frac{v^2}{g}. \end{aligned} \tag{4.3}$$

Thus the height, h, does not depend on the ramp angle or the mass of the ball. It only depends on the initial velocity and the acceleration due to gravity.

Note that some simplifications assumed by the solution are that no kinetic energy is lost when the ball strikes the ramp, and that the heights are actually those of the center of gravity of the ball.

4.2 Hoop, Disk, Cylinder and Sphere

A hoop, a disk, a cylinder and a sphere have the same mass and the same diameter. Each is rolled toward a ramp with the same initial velocity. Which one will reach a higher point on the ramp?

This problem simply involves the conversion of kinetic energy to potential energy. The total kinetic energy when each object is released consists of its forward kinetic energy and its rotational kinetic energy. Hence, for each object

$$P.E. = K.E. = \frac{1}{2} m\, v^2 + \frac{1}{2} I\, \omega^2.$$

where I is the rotational inertia.

The forward kinetic energy for each object is the same, but the rotational kinetic energy depends on the distribution of mass around the center.

The following table shows the values of I and KE_R for several simple shapes.

Shape	Inertia	KE_R
hoop	$m\, r^2$	$m\, v^2/2$
hollow cylinder	$m\, r^2$	$m\, v^2/2$
disk	$m\, r^2/2$	$m\, v^2/4$
solid cylinder	$m\, r^2/2$	$m\, v^2/4$
hollow sphere	$2m\, r^2/3$	$m\, v^2/3$
solid sphere	$2m\, r^2/5$	$m\, v^2/5$

A little thought confirms that I is the same for a hoop and hollow cylinder having equal masses and diameters. Similarly, I is the same for a disk and solid cylinder.

Clearly, the object with the largest rotational inertia will reach the greatest height on the ramp. Given the values for I from the table, the hoop will reach the highest point.

5 Height of Water in Tank

A water tank has sprung a small leak at a point 2 feet from its base on the ground. A second leak, directly over the first, is 5 feet from the base.

A passing physics student noticed that the two streams issuing from the tank were striking the ground at the same spot. He then realized he could calculate the height of the water in the tank. What were his results?

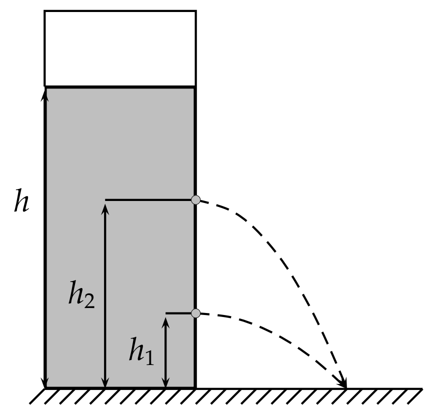

Figure 5.1: Water Tank With Two Leaks

We begin by determining the velocity of the water issuing from the two leaks using Torricelli's theorem. We will use subscripts to link the relevant equations to their respective streams, and insert the known values at the end.

$$v_1^2 = 2g(h - h_1) \tag{5.1}$$
$$v_2^2 = 2g(h - h_2) \tag{5.2}$$

We can find the time required for each stream to strike the ground from the kinematics equation $y = vt_0 + 1/2gt^2$. For this problem $t_0 = 0$.

$$h_1 = \frac{1}{2}gt_1^2 \tag{5.3}$$
$$h_2 = \frac{1}{2}gt_2^2 \tag{5.4}$$

Solving (5.3) and (5.4) for t^2,

$$\begin{aligned} t_1^2 &= \frac{2h_1}{g} \\ t_2^2 &= \frac{2h_2}{g} \end{aligned}$$

The horizontal distance travelled by each stream is vt. We have $v_1 t_1 = v_2 t_2$ or

$$(v_1 t_1)^2 = (v_2 t_2)^2$$

so, substituting from the above

$$\begin{aligned} 2g(h - h_1)\frac{2h_1}{g} &= 2g(h - h_2)\frac{2h_2}{g} \\ (h - h_1)h_1 &= (h - h_2)h_2 \\ hh_1 - h_1^2 &= hh_2 - h_2^2 \\ h(h_1 - h_2) &= h_1^2 - h_2^2 \\ h &= \frac{h_1^2 - h_2^2}{h_1 - h_2} = \frac{(h_1 + h_2)(h_1 - h_2)}{h_2 - h_2} \qquad \text{so,} \\ h &= h_1 + h_2. \end{aligned} \tag{5.5}$$

Given that $h_1 = 2\,\text{ft}$ and $h_2 = 5\,\text{ft}$ the height of the water in the tank $h = 7\,\text{ft}$.

6 Bead Sliding on Wire

In the figure, a vertical hoop supports a wire which is attached from the top of the hoop to any other point. Show that the time required for a frictionless bead to slide down the wire is the same for any destination point. The relevant kinematics equation is

$$x = v_0 t + \frac{1}{2}at^2. \tag{6.1}$$

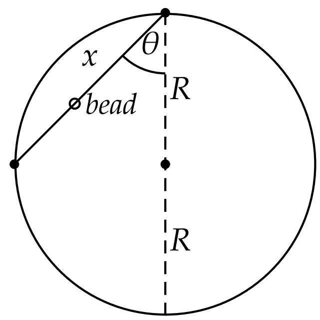

Figure 6.1: Diagram of Bead on Wire

The acceleration is $g\cos\theta$. Since v_0 is zero, we can write

$$x = \frac{1}{2}\cos\theta g t^2. \tag{6.2}$$

But $x = 2R\cos\theta$ from geometry. So

$$2R\cos\theta = \frac{1}{2}\cos\theta g t^2 \tag{6.3}$$

$$2R = \frac{1}{2}g t^2 \quad \text{and,} \tag{6.4}$$

$$t = 2\sqrt{\frac{R}{g}} \quad , \tag{6.5}$$

which is independent of the angle θ, and depends only on the radius of the hoop and the acceleration due to gravity.

Note that the problem and its solution is unchanged if one end of the wire is connected to the bottom of the hoop instead of the top.

Sir James Jeans, in his remarkable book "An Elementary Treatise on Theoretical Mechanics", noted that the solution suggests an interesting minimization problem. Namely, where to place a wire from a fixed point to an inclined plane such that the time for a bead to slide from the point to the plane is a minimum?

The practical form of the solution is to configure a vertical hoop in a plane perpendicular to the ramp with its top at the fixed point P, and to adjust

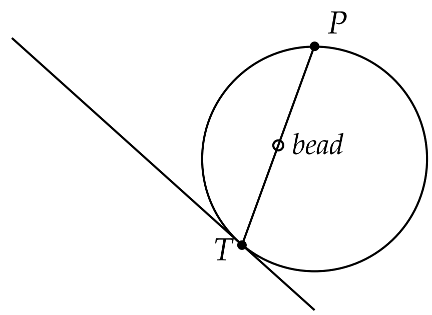

Figure 6.2: Bead Sliding From Point to Plane

its diameter until it just touches the ramp at point T. A wire from the top to the point of tangency will provide the minimal time. Why? Because every other path from the point will touch the hoop at the same time, but the wire chosen is the only one which will have reached the plane in this time.

7 James Bond's Ski Saga

James Bond is skiing down a snowy slope in an attempt to escape a hostile pursuer. Unfortunately, the pursuer has a speed advantage since James is only able to go 25 mph and the pursuer is travelling at 30 mph. Since they are only 1000 feet apart at the beginning, the gap will close in only a few minutes — unless something tips the balance.

James notes that he and his pursuer carry the same kind of rifle and estimates that their masses are about the same. Recalling his elementary physics, he realizes that each time he fires his weapon back at the pursuer, his forward momentum and velocity will increase. On the other hand, when the pursuer fires *his* momentum and velocity will decrease.

Every time James fires a round, his adversary fires back. We would like to know how many rounds James must fire to assure that his pursuer cannot catch up with him. Assume that all rounds miss their targets (otherwise this exercise would terminate).

Let M be the mass of each, including the man, skis, weapon, backpack, etc. The mass of each bullet is m, and the muzzle velocity is v. The governing equation is then

$$\begin{aligned} 30\,M + n\,mv &= 35\,M - n\,mv \qquad \text{so,} \\ 2n\,mv &= 5\,M \end{aligned} \tag{7.1}$$

where n is the number of rounds fired.

We would like to solve the problem using the *cgs* system of units. Then 5 mph = 223 cps.

Now let $M = 100,000$ g, $m = 15$ g and $v = 30,000$ cps. Substituting in (7.1), we must solve

$$\begin{aligned} n &= \frac{223 \cdot 10^5}{2 \cdot 15 \cdot 3 \times 10^4} \\ n &\approx 25. \end{aligned} \tag{7.2}$$

Hence, when each man has fired 25 rounds, the gap between them will stop decreasing and begin to increase, assuring James' escape.

8 Moment of Inertia

In this section we derive formulae for determining the *moment of inertia* or *second moment* of a mass around an axis for several common physical shapes.

The moment of inertia is evaluated by summing the products of all mass elements by the squared moment arm associated with the element.

8.1 Constant Moment Arm

The simplest case is that of a point mass, and the moment of inertia can be immediately written as

$$I = mR^2,$$

where I is the moment of inertia, m is the mass of the object and R is the distance from the axis to the object. It is worth noting that this relation also holds for a thin hoop or ring with the axis perpendicular to the object and through its center. By extension it also applies to a thin cylindrical shell with the axis of rotation coincident with the axis of the cylinder. All these shapes have essentially the same moment arm.

8.2 Moment of Disk or Solid Cylinder About Axis

We assume the axis of rotation is perpendicular to the disk and through its center, and coincident with the axis of the cylinder.

Let the radius of the disk be R, and a mass element $\delta m = \sigma\delta S$. A surface element can be defined as $\delta S = r\,d\theta\,dr$. Substituting the surface element

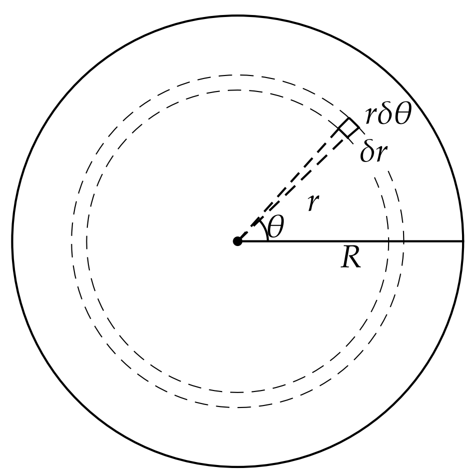

Figure 8.1: Surface Element for Disk

into the equation for a mass element, we have $\delta m = \sigma r\,dr\,d\theta$. Since an element of inertia can be expressed $\delta I = \delta m r^2$, we can write

$$\begin{aligned} \delta I &= \sigma r^2 r\delta r\,\delta\theta \text{ and,} \\ \delta I &= \sigma r^3\,\delta r\,\delta\theta \qquad (8.1) \end{aligned}$$

Summing the elements and taking limits gives

$$I = \sigma \int_0^{2\pi} \int_0^R r^3 \, dr \, d\theta.$$

Performing the integrations,

$$\begin{aligned} I &= \sigma 2\pi \frac{R^4}{4} \\ I &= \sigma \pi R^4/2. \end{aligned} \tag{8.2}$$

The mass of the disk is $\sigma\pi R^2$. Substituting in 8.2,

$$I = \frac{1}{2} m R^2.$$

8.3 Moment of Thin Spherical Shell About Axis

This problem can be set up in spherical coordinates so that conversions from Cartesian coordinates are not required.

The mass element for this case is assumed to exist on the surface of a sphere. An element of the surface area of a sphere, δS, is related to an element of mass by $\delta m = \sigma \delta S$, where σ is the mass per unit area. In the figure, $\rho = R \sin\theta$ is the length of the moment arm for the mass element. Hence, $\delta I = \rho^2 \delta m$. The area of the surface element is $R\delta\theta \times R \sin\theta\delta\phi$ or $R^2 \sin\theta\delta\theta\delta\phi$.

Expanding, we have

$$\begin{aligned} \delta I &= (R \sin\theta)^2 \, \delta m \\ \delta I &= R^2 \sin^2 \theta \sigma \, \delta S \\ \delta I &= R^2 \sin^2 \theta \sigma R^2 \sin\theta \, \delta\theta \, \delta\phi \\ \delta I &= R^4 \sigma \sin^3 \theta \, \delta\theta \, \delta\phi \end{aligned}$$

Summing the increments and taking limits, we may write the following integral:

$$I = R^4 \sigma \int_0^{2\pi} \int_0^{\pi} \sin^3 \, d\theta \, d\phi$$

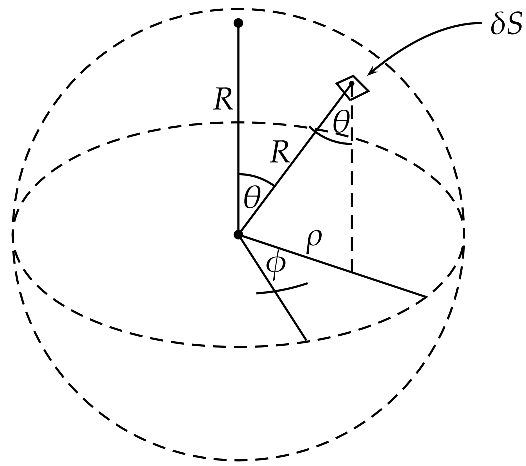

Figure 8.2: Thin Shell Diagram

It is easiest to integrate with respect to ϕ first.

$$I = R^4 \sigma 2\pi \int_0^{\pi} \sin^3\theta \, d\theta$$

To solve this integral, recall that $\sin\theta \, d\theta = d(-\cos\theta)$. Making the substitution, mindful that changing the variable of integration requires changing the integration limits as follows,

$$\theta \Big|_0^{\pi} : \ \cos\theta \Big|_1^{-1}$$

we can write,

$$\begin{aligned} I &= 2\pi R^4 \sigma \int_1^{-1} \sin^2\theta \, d(-\cos\theta) \\ I &= 2\pi R^4 \sigma \int_1^{-1} (1 - \cos^2\theta) \, d(-\cos\theta) \\ I &= 2\pi R^4 \sigma \int_1^{-1} (\cos^2\theta - 1) \, d(\cos\theta). \end{aligned} \tag{8.3}$$

It may be convenient to replace $\cos\theta$ in 8.3 with a simpler variable, say x.

We now have,

$$
\begin{aligned}
I &= 2\pi R^4\sigma \int_1^{-1} (x^2 - 1)\, dx \\
I &= 2\pi R^4\sigma \left| \frac{x^3}{3} - x \right|_1^{-1} \\
I &= 2\pi R^4\sigma \Big| (-1/3 + 1) - (1/3 - 1) \Big| \\
I &= 2\pi R^4\sigma (4/3) \\
I &= \frac{8}{3}\pi R^4\sigma. \qquad (8.4)
\end{aligned}
$$

Noting that the total mass, m, is $\sigma 4\pi R^2$, we can reduce (8.4) to

$$I = \frac{2}{3} m R^2. \qquad (8.5)$$

8.4 Moment of Solid Sphere About Axis

The moment of enertia for a homogeneous, solid sphere about an axis can be found by integrating spherical shells, by integrating disks, or by solving the equations for the moment of an element of mass throughout the volume. We will use the latter.

The moment of a mass element is $I = \int_V l^2\, dm$, where $dm = \rho\, dV$, l is the distance of the element from the axis and ρ is the mass density. Note that the distance to an element from the center of the sphere is r and $l = r \sin\theta$ where θ is the angle between the axis and the radius. The volume element, dV, is

$$dV = (2\pi\, l)(dr)(r\, d\theta) = 2\pi\, r^2\, \sin\theta\, dr\, d\theta. \qquad (8.6)$$

Since

$$\rho = \frac{m}{V} = \frac{m}{\frac{4}{3}\pi\, R^3}$$

and $dm = \rho dv$, the moment of inertia is

$$I = \int l^2\, dm = \int (r\, \sin\theta)^2 \rho\, dV.$$

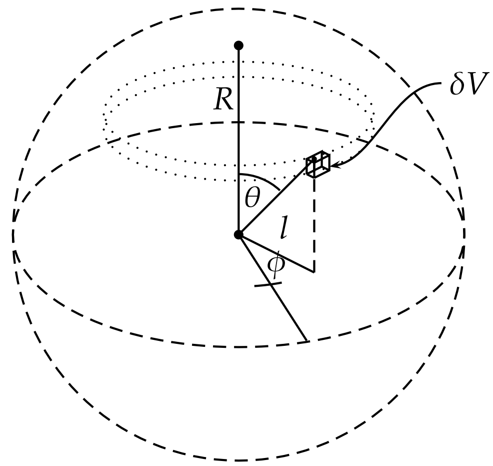

Figure 8.3: Volume Element Diagram

For our sphere

$$
\begin{aligned}
I &= \int_{\theta=0}^{\pi}\int_{r=0}^{R}(r^2\,\sin^2\theta)\left(\frac{m}{\frac{4}{3}\pi\,R^3}\right)(2\pi\,r^2\sin\theta)\,dr\,d\theta \\
I &= \int_{\theta=0}^{\pi}\int_{r=0}^{R}\frac{3m}{2R^3}r^4\,\sin^3\theta\,dr\,d\theta \\
I &= \int_{\theta=0}^{\pi}\frac{3m}{2R^3}\left[\frac{r^5}{5}\sin^3\theta\right]_{r=0}^{R}\,d\theta \\
I &= \frac{3}{10}m\,R^2\int_{\theta=0}^{\pi}\sin^3\theta\,d\theta \\
I &= \frac{3}{10}m\,R^2\int_{\theta=0}^{\pi}\sin\theta(1-\cos^2\theta)\,d\theta. \qquad (8.7)
\end{aligned}
$$

It is convenient to change the variable of integration from θ to $-\cos\theta$.

Let $x = -\cos\theta$. Then the limits of the above integral become

$$\theta\Big|_0^{\pi} : \; x\Big|_{-1}^{1}.$$

The corresponding integral is

$$I = \frac{3}{10} m R^2 \int_{-1}^{1} (1 - x^2)\, dx$$
$$I = \frac{3}{10} m R^2 \left[x - \frac{x^3}{3} \right]_{-1}^{1}$$
$$I = \frac{3}{10} m R^2 ((1 - 1/3) - (-1 + 1/3))$$
$$I = \frac{3}{10} m R^2 \left(\frac{4}{3} \right)$$

Finally,

$$I = \frac{2}{5} m R^2. \tag{8.8}$$

9 Vertical Loop

Here are a few problems involving the transformation of kinetic energy to potential energy and vice versa.

9.1 Ball on String

A small ball at the end of string is swung in a circular vertical loop. The speed of rotation is decreased to the point that the tension on the string at the top of the loop drops to zero. Analyze the system for this condition.

The forces on the ball consist of its weight, the centrifugal force due to motion along a circular path and the tension from the string which provides the centripetal force.

$$F_t = \frac{mv^2}{l} - mg - T_s, \tag{9.1}$$

where F_t is the sum of the forces, m is the mass of the ball, l is the length of the string and T_s is the tension. At the top of the loop, the forces are in

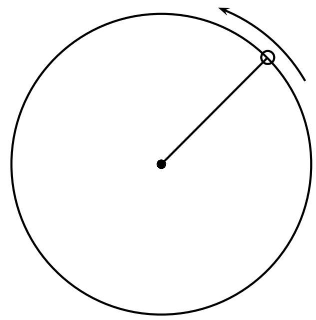

Figure 9.1: Ball Swung in Vertical Loop

equilibrium so $F_t = 0$. Let v_t be the velocity of the ball at the top. If the tension drops to zero there we have

$$
\begin{aligned}
\frac{mv_t^2}{l} &= mg, \qquad \text{so,} &(9.2)\\
v_t^2 &= gl \qquad \text{and,} &(9.3)\\
v_t &= \sqrt{gl}. &(9.4)
\end{aligned}
$$

The kinetic energy of the ball at the top of the loop is

$$
\frac{mv_t^2}{2}. \qquad (9.5)
$$

At the bottom of the loop, the kinetic energy is increased by the potential energy at the top. From this we can determine the velocity at the bottom

$$
\frac{mv_b^2}{2} = \frac{mv_t^2}{2} + 2mgl \qquad (9.6)
$$

Solving for v_b:

$$
\begin{aligned}
v_b^2 &= v_t^2 + 4gl &(9.7)\\
v_b^2 &= gl + 4gl = 5gl &(9.8)\\
v_b &= \sqrt{5gl}. &(9.9)
\end{aligned}
$$

The total force on the ball at the bottom is the sum of the centrifugal force and its weight. This determines the tension on the string.

$$F_b = \frac{mv_b^2}{l} + mg \tag{9.10}$$

$$F_b = \frac{5mgl}{l} + mg \tag{9.11}$$

$$F_b = 6mg. \tag{9.12}$$

So when the rotation rate is such that the ball experiences no vertical forces at the top of the loop, it experiences 6 g's at the bottom.

9.2 Cart on Track

Here we have a cart on a track which consists of a vertical circular loop. Of course we do not want the cart to fall off the track at the top of the loop, so it must have sufficient forward velocity that its centrifugal force keeps it in contact. We wish to find the height from which the cart must be dropped on the leading ramp to satisfy this requirement.

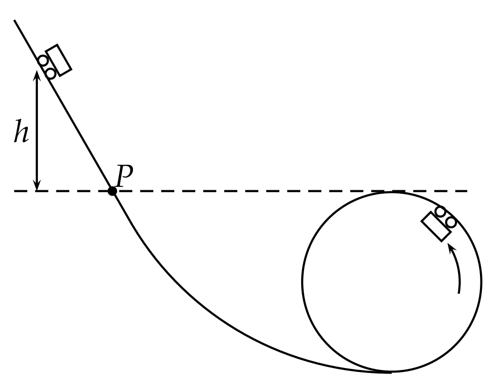

Figure 9.2: Cart Rolling Around Loop

We know from the previous problem, that the velocity of the cart at the top of the loop must be

$$v_t = \sqrt{gR} \tag{9.13}$$

where R is the radius of the loop, and g is the acceleration due to gravity. Since we are assuming that friction is negligible, the kinetic energy at point P must the same as the kinetic energy at the top of the loop. This energy must be provided by the conversion of potential energy to kinetic energy from the point of release of the cart to point P.

So $mgh = mv_t^2/2$ or

$$h = \frac{v_t^2}{2g}.$$

Substituting from Eq. (9.13),

$$h = \frac{R}{2}.$$

This is the height above point P from which the cart must be dropped. The total height above ground is $R/2 + 2R = 5R/2$.

9.3 Pole Vault

A pole vaulter performs the remarkable feat of converting his forward kinetic energy to vertical potential energy. Using this information, we can estimate the maximum height possible for a pole vault. We assume (not quite correctly) that the pole itself cannot store and release energy during the vault. Also assume that the conversion is lossless.

Suppose the vaulter can run at 20.5 mph. This is about 30 fps. Then his kinetic energy at the start of the vault is $KE = mv^2/2 = 450\,m$. His potential energy at the top of the vault is $PE = mgh = 32m\,h$. So

$$h = 450/32\,\text{ft} = 14\,\text{ft}.$$

But this height refers to the height of the center of gravity of the vaulter. At the start of the vault, his center of gravity is about 3.5 ft above the ground. When he clears the bar, it is about 5 in = 0.41 ft above the bar.[1] Hence the

[1] It has been argued that a vaulter may clear the bar with his center of gravity *below* the bar.

maximum height of the bar must be $3.5 + 14 - 0.41\,\text{ft} = 17.1\,\text{ft}$ above the ground.

This height was actually reached during the 1960s and established a world record at the time. By 1991 the record had soared to over 20 ft, a height which was achieved by Ukranian athlete, *Sergei Bubka*. The increased heights are largely due to improvements in the pole, which allow the vaulter to store energy in the pole by flexing it just before the jump. This energy is returned during the jump to contribute to the overall height.

It is clear the a taller pole vaulter has an advantage over his shorter competitors by the increased height of his center of gravity. Perhaps in the future a handicap system would be appropriate.

10 Cue Ball Slip Problems

10.1 Slip Problem #1

A cue ball is struck along a line through its center and parallel to the table. It moves forward initially with zero angular rotation, sliding across the felt, but eventually rolls without slipping. How far does it travel before pure rolling motion occurs?

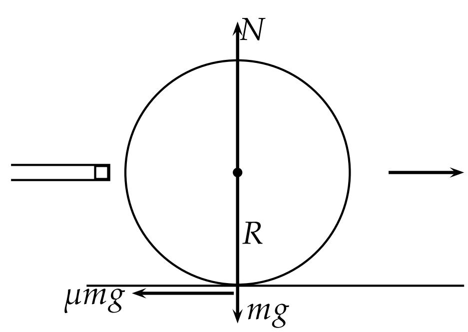

Figure 10.1: Cue Ball Motion Diagram

This interesting problem yields to elementary linear and rotational kinematics. It's worth making a few preliminary observations about the problem.

First, the initial linear velocity imparted to the cue ball, $v_{0+} = v$, is a maximum at the moment of impact. During the course of travel the velocity will decrease due to the frictional drag exerted by the table felt on the ball. At the same time, a torque will be exerted on the ball by this same frictional force. Although the problem does not require consideration of kinetic energy, it is clear that the initial kinetic energy is purely linear and when slippage stops the resulting kinetic energy is distributed between linear kinetic energy and rotational kinetic energy.

The normal force, N, at any time, is simply due to gravity and is mg. The frictional force due to drag is then μmg, where μ is the coefficient of friction. The drag is responsible for the only acceleration on the cue ball.

The velocity at any time is

$$v_t = v + at = v - \mu g t. \tag{10.1}$$

The torque, τ, is μmgR. But $\tau = I\alpha$, where I is the moment of inertia and α is the angular acceleration. Since it is known that the moment of inertia of a solid sphere about its center is

$$\frac{2}{5}mr^2,$$

we can solve for the angular acceleration.

$$\alpha = \frac{\tau}{I} = \frac{\mu mgR}{\frac{2}{5}mR^2} = \frac{5}{2}\frac{\mu g}{R}$$

The angular velocity is $\omega_t = \omega_0 + \alpha t$ where ω_0 is zero. Thus,

$$\omega_t = \alpha t = \frac{5}{2}\frac{\mu g}{R}t. \tag{10.2}$$

Pure rolling motion occurs when $v_t = R\omega_t$. Substituting from (10.1) and

(10.2) and solving for t,

$$\begin{aligned} v - \mu g t &= R \frac{5}{2} \frac{\mu g t}{R} = \frac{5}{2} \mu g t \\ v &= \frac{7}{2} \mu g t, \qquad \text{so,} \\ t &= \frac{2}{7} \frac{v}{\mu g}. \end{aligned}$$

We can now find the distance from $d = vt + \frac{1}{2}at$.

$$\begin{aligned} d &= vt - \mu g t^2 \\ d &= v\left(\frac{2}{7}\frac{v}{\mu g}\right) - \frac{1}{2}\mu g \left(\frac{2}{7}\frac{v}{\mu g}\right)^2 \\ d &= \frac{2}{7}\frac{v^2}{\mu g} - \frac{2}{49}\frac{v^2}{\mu g} \\ d &= \frac{12}{49}\frac{v^2}{\mu g} \end{aligned} \tag{10.3}$$

10.2 Slip Problem #2

At what point should a cue ball be struck so that it immediately rolls with no slipping?

The objective here is to impart a rotational velocity as well as a linear velocity such that the equation

$$v = \omega R \tag{10.4}$$

is satisfied.

This problem can be recast in the following form: At what point should the cue ball be struck so that the ball rotates around its point of contact with the table? The condition is valid at the moment of impact even though subsequent movement of the ball will be constrained by the table surface.

We begin by finding the moment of inertia of the ball around the point of contact. Using the parallel axis theorem, $I_p = I_g + mk^2$, where I_g is the

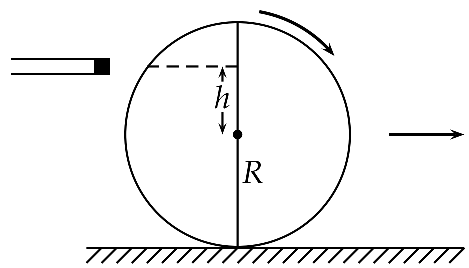

Figure 10.2: Cue Ball Motion Diagram #2

moment of inertia around the center of mass and k is the distance from the center of mass to the new point of rotation. This new point is one radius away from the center.

$$I_p = \frac{2}{5}mR^2 + mR^2$$
$$I_p = \frac{7}{5}mR^2 \qquad (10.5)$$
$$(10.6)$$

The impulse at the moment of impact results in a change of momentum $F' = mv$. Note that $v_0 = 0$. The corresponding change in angular momentum is $F' \cdot (R + h) = I_p\,\omega$. We now have, substituting from (10.4),

$$mv(R + h) = \frac{7}{5}mR^2\frac{v}{R}$$
$$R + h = \frac{7}{5}R$$
$$h = \frac{2}{5}R. \qquad (10.7)$$

11 Object on a Bowling Ball

11.1 Bug on a Bowling Ball

A bug sitting on top of a bowling ball begins to slide off with negligible friction. Determine the angle at which the bug leaves the surface.

This problem is easily solved with the aid of the diagram in Fig. (11.1). The

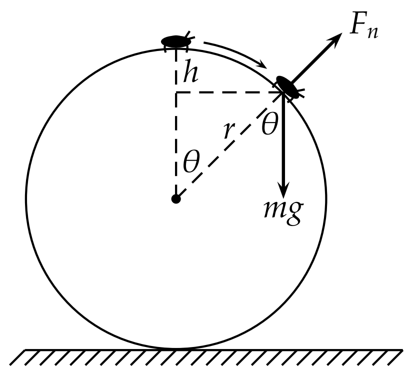

Figure 11.1: Bug on a Bowling Ball

forces on the bug include the *centripetal* force due to gravity, the *centrifugal* force due to motion along the curve and the resulting *normal* force.

$$F_n = \frac{mv^2}{r} - mg\cos\theta \tag{11.1}$$

The gain in kinetic energy as the bug slides is provided by the loss in potential energy. Since $h = r - r\cos\theta$ we have,

$$\begin{aligned} \frac{mv^2}{2} &= mgh \\ \frac{mv^2}{2} &= mgr(1-\cos\theta) \\ \frac{mv^2}{r} &= 2mgr(1-\cos\theta) \end{aligned} \tag{11.2}$$

At the moment the bug leaves the ball, the normal force F_n becomes zero. We can now substitute the value of the centrifugal force from 11.1 into 11.2.

$$\begin{aligned}
mg\cos\theta &= 2mgr(1-\cos\theta) \\
\cos\theta &= 2-2\cos\theta \\
\cos\theta &= 2/3 \\
\theta &= \arccos(2/3) \qquad \text{and finally,} \\
\theta &= 48.2\ \text{degrees.}
\end{aligned} \tag{11.3}$$

11.2 Marble on a Bowling Ball

This problem examines the behavior of a marble as it rolls without slipping from the top of a bowling ball to the point at which it leaves the surface. In Fig. (11.2) R is the radius of the bowling ball and r is the radius of the marble.

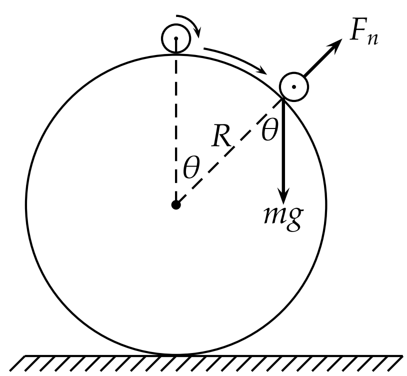

Figure 11.2: Marble on a Bowling Ball

The initial conditions require that the linear and rotational kinetic energies are zero. When the marble leaves the surface of the bowling ball, the sum of these energies must equal the loss in potential energy. Since the marble rolls without slipping, at any instant $v = R\omega$.

We begin by finding an expression for the rotational kinetic energy in terms of v.

$$\begin{aligned} K.E._{rotational} &= \frac{1}{2}\left(\frac{2}{5}mR^2\right)\omega^2 \\ &= \frac{1}{5}mv^2 \end{aligned} \tag{11.4}$$

The total kinetic energy is

$$K.E._{total} = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 = \frac{7}{10}mv^2.$$

The initial potential energy is $mg(R+r)$ and at the point of departure it is $mg(R+r)\cos\theta$. The loss in potential energy is $mg(R+r)(1-\cos\theta)$, so

$$\begin{aligned} mg(R+r)(1-\cos\theta) &= \frac{7}{10}mv^2 \\ g(R+r)(1-\cos\theta) &= \frac{7}{10}v^2 \\ v^2 &= \frac{10}{7}g(R+r)(1-\cos\theta) \end{aligned} \tag{11.5}$$

The normal force on the marble is $mg\cos\theta - mv^2/(R+r)$ and at the point of departure this becomes zero so,

$$\begin{aligned} mg\cos\theta &= \frac{mv^2}{R+r} \\ \cos\theta &= \frac{v^2}{(R+r)g} \\ v^2 &= \cos\theta(R+r)g. \end{aligned} \tag{11.6}$$

Substituting for v^2 in (11.5) and (11.6),

$$\begin{aligned}
\cos\theta(R+r)g &= \frac{10}{7}(1-\cos\theta)(R+r)g \\
\cos\theta &= \frac{10}{7}(1-\cos\theta) \\
\cos\theta + \frac{10}{7}\cos\theta &= \frac{10}{7} \\
\frac{17}{7}\cos\theta &= \frac{10}{7} \\
\cos\theta &= \frac{10}{17} \\
\theta &= \arccos\left(\frac{10}{17}\right) \qquad \text{and finally,} \\
\theta &\approx 54.0\,\text{degrees.} \qquad (11.7)
\end{aligned}$$

12 Volume of Solid Ring

A solid sphere is bored out such that the radial axis of the removed cylinder passes through the center. The ring of remaining material stands 6 centimeters high. What is the volume of this ring?

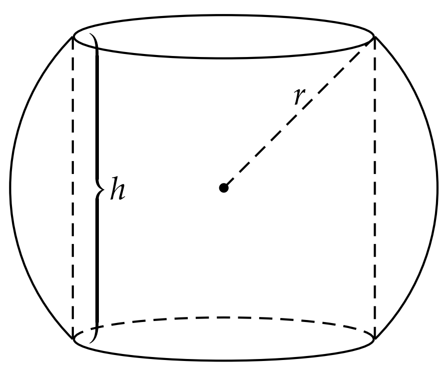

Figure 12.1: Sphere with Cylindrical Bore

One way to compute the volume of the ring is to subtract the volume of the removed material from the volume of the original sphere. This bored out material can be regarded as a right cylinder with spherical end caps on the two flat surfaces.

Another way is to compute the volume of the partial sphere, excluding the end caps, and then subtract the volume of the right cylinder. We will choose this method.

Referring to Fig. (12.1), the volume of the partial sphere can be computed by using the disk method and integrating from the top edge of the ring to the bottom edge. Using the center of the sphere as the origin, x as the horizontal axis through the center and y the vertical axis, the equation is

$$\begin{aligned} V_s &= \int_{-3}^{+3} \pi\, x^2\, dy \\ &= \int_{-3}^{+3} \pi(R^2 - y^2)\, dy. \end{aligned} \tag{12.1}$$

The right cylinder has volume

$$\begin{aligned} V_c &= 6\pi r^2 \\ &= 6\pi(R^2 - 9) \end{aligned} \tag{12.2}$$

where r is the radius of the cylinder. Note the requirement: $R \geq 3$.

So the volume of the ring is $V = V_s - V_c$. Subtracting (12.2) from (12.1)

$$\begin{aligned} V &= \pi \int_{-3}^{+3} (R^2 - y^2)\, dy - 6\pi(R^2 - 9) \\ &= \pi \left[R^2 y - \frac{y^3}{3} \right]_{-3}^{+3} - 6\pi R^2 + 54\pi \\ &= \pi(3R^2 - 9 + 3R^2 - 9) - 6\pi R^2 + 54\pi \\ &= 36\pi \end{aligned} \tag{12.3}$$

Surprisingly, this result is independent of the radius of the sphere. As long as the radius $R \geq 3$ the result holds. Hence, another way to compute the volume of the ring is to compute the volume of a sphere with $R = 3$

representing the case of an infinitesimal bored out volume. This sphere, of course, has volume given by

$$V = \frac{4}{3}\pi R^3 = 36\pi$$

13 Orbital Velocity for Low Earth Orbit

Neglecting air friction, an object will maintain a low altitude orbit when the centrifugal force due to its motion in a circular orbit is equal to the gravitational force attracting it to earth.

Note that for this situation, the *centripetal* force is provided by gravitation, attracting the object toward the earth's center. The reactive force, directed away from the earth is *centrifugal*.

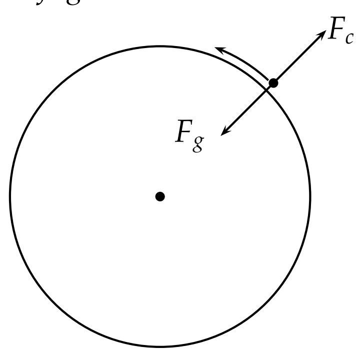

Figure 13.1: Force Diagram for Object in Orbit

The centrifugal force is

$$F_c = \frac{mv^2}{r},$$

where r is the radius of the orbit, m is the mass of the object and v is the velocity. The orbital radius is assumed approximately equal to the earth's radius for low orbit. The gravitational force is

$$F_g = -mg, \tag{13.1}$$

where g is the acceleration due to gravity. We are again assuming low orbit.

Then $F_c + F_g = 0$ and

$$\begin{aligned} \frac{mv^2}{r} &= mg \\ v^2 &= gr, \qquad \text{and,} \\ v &= \sqrt{gr}. \end{aligned} \tag{13.2}$$

Assuming earth's radius to be 3960 miles, and the acceleration due to gravity is 32 feet per second^2, we have

$$\begin{aligned} v &= \sqrt{32/5280 \times 3960} \\ v &= 4.9 \quad \text{mi/sec.} \end{aligned} \tag{13.3}$$

14 Escape Velocity

An object will overcome the force of gravity when its kinetic energy in the direction away from earth exceeds its potential energy. We find the minimum kinetic energy required by solving $K.E. = P.E.$.

Then

$$\frac{mv^2}{2} = mgr,$$

where m is the mass of the object, v is its velocity, g is the acceleration due to gravity and r is the radius of the earth.

Solving for v

$$\begin{aligned} v^2 &= 2gr \\ v &= \sqrt{2gr} \end{aligned} \tag{14.1}$$

Taking the earth's radius as 3960 miles and the acceleration due to gravity as 32 feet per second^2, we have

$$v = \sqrt{2 \times 32/5280 \times 3960} \tag{14.2}$$

which reduces to 6.9 miles per second.

15 Geosynchronous Orbit

Communications satellites can be placed in equatorial orbits at a distance which results in an orbital period of one day. Thus the satellite occupies a stationary position above the surface of the earth. To determine the height of this orbit, we simply equate th centripetal force due to gravity with the centrifugal force resulting from motion along the circular path.

Using M_e for the mass of the earth, m_s for the mass of the satellite, G for the gravitational constant, ω for the angular velocity and h for the height of the orbit above the earth's surface, we have

$$\frac{GM_e m_s}{(R_e + h)^2} = m_s \omega^2 (R_e + h)$$

Then

$$\begin{aligned}
\frac{GM_e}{(R_e + h)^2} &= \omega^2 (R_e + h) \\
(R_e + h)^3 &= \frac{GM_e}{\omega^2} \\
R_e + h &= \left(\frac{GM_e}{\omega^2}\right)^{1/3} \\
R_e + h &= \left(\frac{(6.67 \times 10^{-11}) \times (5.98 \times 10^{24})}{(7.27 \times 10^{-5})^2}\right)^{1/3} \\
R_e + h &= 4.23 \times 10^7 \\
h &= 4.23 \times 10^7 - 6.37 \times 10^6 \\
h &= 3.59 \times 10^7 \text{ (meters).}
\end{aligned} \tag{15.1}$$

This is 22,300 mi. and amounts to about 5.6 earth radii.

16 Simple Harmonic Motion

Harmonic motion is considered simple if it is undamped, *i.e.* if it continues to oscillate uniformly over time.

Of particular interest are the *frequency*, f, or *period*, T of the oscillations.

Consider an object subject to only two forces: one due to the acceleration of the object and the other due to a restoring force. The total force is

$$F = ma + kx,$$

where k is the spring constant or restoring force.

This can be expressed as a second order linear differential equation as follows,

$$\begin{aligned} m\frac{d^2x}{dt^2} + kx &= 0 \\ \frac{d^2x}{dt^2} + \frac{k}{m}x &= 0. \end{aligned} \tag{16.1}$$

It is known that solutions to such equations involve trig functions. We define a generalized cosine as, $x = A\cos(\omega t + \phi)$. Then, $dx/dt = -A\omega\sin(\omega t + \phi)$ and $d^2/dt^2 = -A\omega^2\cos(\omega t + \phi)$. Substituting in Ref. (16.1),

$$-A\omega^2\cos(\omega t + \phi) + A\frac{k}{m}\cos(\omega t + \phi) = 0 \tag{16.2}$$

So,

$$\begin{aligned} \omega^2 &= \frac{k}{m} \\ \omega &= \sqrt{\frac{k}{m}}. \end{aligned} \tag{16.3}$$

But $\omega = 2\pi f = 2\pi/T$. Hence,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{and} \tag{16.4}$$

$$T = 2\pi\sqrt{\frac{m}{k}}. \tag{16.5}$$

17 Gravitational Field Inside a Spherical Shell

In this problem we prove that the gravitational field inside a thin spherical shell of finite mass is zero. By extension, the field inside a thick shell whose inner and outer radii are finite is also zero. In this figure, R is the radius of

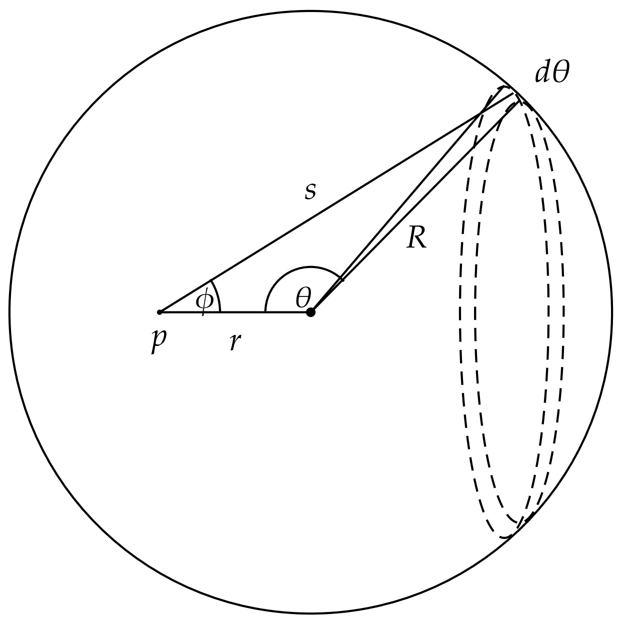

Figure 17.1: Field Inside a Thin Spherical Shell

the shell and M its mass. The mass per unit area is $\sigma = M/S$, where S is the surface area. An object of mass m is located at point p, which is at distance r from the center of the sphere.

An element of mass is given by $dM = \sigma \times 2\pi R \times R \sin\theta \, d\theta$, which corresponds to the ring on the surface. If the gravitational force along the line s is resolved into inline and perpendicular components, we find that the perpendicular components ($R \sin\theta$) cancel and only the inline ($R \cos\theta$) components contribute. Hence the acceleration due to a gravitational element is:

$$da = \frac{G \, dM}{s^2} \cos\phi.$$

Substituting for *dM* and integrating over the surface, we have

$$a = \sigma 2\pi\, G \int_{\theta=0}^{\pi} \frac{\cos\phi\ \sin\theta}{s^2} d\theta.$$

Now we express s and ϕ in terms of θ.

$$s^2 = R^2 + r^2 - 2R\,r\cos\theta$$

using the *law of cosines*. Differentiating,

$$\begin{aligned} 2s\,ds &= 2R\,r\ \sin\theta\,d\theta \\ \sin\theta\,d\theta &= \frac{s\,ds}{R\,r} \end{aligned} \tag{17.1}$$

For angle ϕ

$$\begin{aligned} R^2 &= r^2 + s^2 - 2r\ \cos\phi \qquad \text{and} \\ \cos\phi &= \frac{r^2 + s^2 - R^2}{2r\,s}. \end{aligned} \tag{17.2}$$

We can now make the necessary substitutions, mindful that changing the variable of integration changes the limits.

$$\theta\Big|_0^{\pi} : s\Big|_{R-r}^{R+r}$$

$$\begin{aligned} a &= -2\pi\,G\,\sigma\,R^2 \int_{s=R-r}^{s=R+r} \frac{1}{s^2}\frac{r^2+s^2-R^2}{2\,r\,s}\frac{s\,ds}{R\,r} \\ a &= -\frac{\pi\,G\,\sigma\,R}{r^2}\int_{s=R-r}^{s=R+r}\left(1+\frac{r^2-R^2}{s^2}\right)ds \end{aligned} \tag{17.3}$$

Now $\sigma = M/4\pi\,R^2$, so

$$a = -\frac{G\,M}{4\,R\,r^2}\int_{s=R-r}^{s=R+r}\left(1+\frac{r^2-R^2}{s^2}\right)ds.$$

Integrating,

$$
\begin{aligned}
a &= -\frac{GM}{4R\,r^2}\left[s - \frac{r^2 - R^2}{s}\right]_{s=R-r}^{s=R+r} \\
a &= -\frac{GM}{4R\,r^2}\left[(R+r) - (R-r) - (r^2 - R^2)\left(\frac{1}{R+r} - \frac{1}{R-r}\right)\right] \\
a &= -\frac{GM}{4R\,r^2}\left[2r - (r^2 - R^2)\left(\frac{(R-r) - (R+r)}{R^2 - r^2}\right)\right] \\
a &= -\frac{GM}{4R\,r^2}\left[2r + (-2r)\right] \\
a &= 0. \qquad (17.4)
\end{aligned}
$$

Hence, the acceleration due to gravity at any point inside a thin spherical shell is identically zero!

There are other, simpler ways to find this solution. For example, *Gauss' Law* immediately yields the same result.

18 Tunnel Through the Earth

Suppose a straight tunnel is cut from the surface of the earth to the opposite side through the center.

An object is dropped down the tunnel and, assuming that air friction is negligible, we wish to determine the time it takes for the object to return to its starting point.

The only force acting on the object is gravitational. From previous problems we know that only that portion of the earth's mass contained in the sphere with radius equal to the object's height above the center will contribute.

let

$$\sigma = \frac{M}{V}$$

and

$$V = \frac{4}{3}\pi R^3$$

where M is the mass of the earth and V is the volume. be the mass density of the earth. Assuming uniform density at any radius from the center

$$\sigma = \frac{M_r}{V_r}.$$

The governing equation for an object at any distance r from the center is

$$\begin{aligned} ma_r &= \frac{G\,m\,M_r}{r^2} \\ a_r &= \frac{G\,M_r}{r^2} \\ a_r &= \frac{4\pi\,G\,\sigma\,r^3}{3r^2} \\ a_r &= \frac{4}{3}\pi\,G\,\sigma\,r \qquad (18.1) \end{aligned}$$

The force, F is

$$F = ma_r = \frac{4}{3}\pi\,G\,\sigma\,m\,r.$$

But this corresponds to the differential equation

$$\frac{d^2 r}{dt^2} - k\,r = 0$$

whose solution is the same as that of an object subject to a spring force without friction. That is, *simple harmonic motion.* The period is given by

$$T = 2\,\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{\sqrt{\frac{4}{3}\pi\,\sigma\,G}}.$$

At the surface, the gravitional force is the weight of the object

$$W = mg = \frac{G\,M\,m}{R^2} = \frac{4}{3}\pi\,\sigma\,G\,m\,R.$$

Thus

$$\frac{g}{R} = \frac{4}{3}\pi\,\sigma\,G,$$

so

$$T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{6370 \times 10^3\,m}{9.8\,m/\mathrm{sec}^2}} = 84.8 \quad \text{min.}$$

19 Snell's Law

It is believed that Snell developed his famous equation by purely empirical means. He made numerous measurements of the refracting properties of various materials and found a relationship which made accurate predictions. Later, is was found that his result could be proven.

This proof of Snell's law is purely geometric and only requires the initial assumption that the *index of refraction*, n, is related to the speed of light in the media by the following relation:

$$n = \frac{c}{v}$$

where v is the speed of light in the medium. The geometry is illustrated in Fig. (19.1), with two parallel rays approaching the interface at an angle, θ_i, from the normal. Media $m1$ and $m2$ have indices of refraction, n_i and n_r, respectively.

Snell's law is usually stated as,

$$n_i \sin\theta_i = n_r \sin\theta_r,$$

where i is an incident ray and r is a refracted ray.

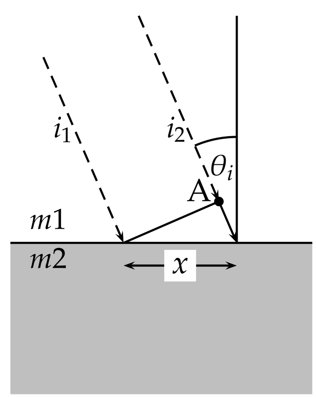

Figure 19.1: Plane Wave Incident on Interface

If the two rays shown travel together, then i_2 just reaches point 'A' when i_1 strikes the interface. i_2 completes the remaining journey to the interface at velocity c/n_i and covers distance $x \sin\theta_i$.

Fig. (19.2) shows the incident and refracted rays with critical points labelled. While i_2 completes its journey to the interface, i_1 is refracted into

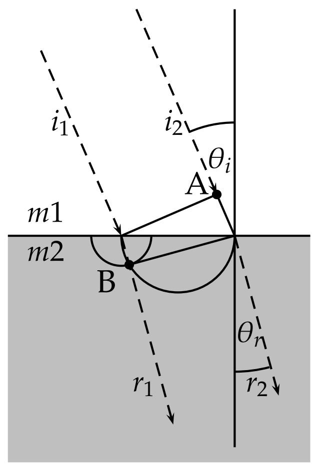

Figure 19.2: Refracted Wave in Medium

$m2$ travelling at the new velocity c/n_r. It reaches point 'B' when i_2 reaches the interface. Note that the acute angle formed by the entry point of i_1 and the right triangle at 'A' is θ_i, and the corresponding acute angle formed by the entry point of r_2 and the right triangle at 'B' is θ_r.

Point 'B' is found by the intersection of two circles. One is the circle centered at the entry point of i_1 at the interface and with radius equal to the distance travelled in medium m_2 while i_2 travels its excess distance to the interface in m_1. The second circle is centered halfway between the entry point of i_1 and i_2 along the interface, and with radius equal to half that distance $x/2$. This is the locus of right triangles with x as a hypotenuse.

The ratio of the two distances is the same as the ratio of the sines of angles θ_i/θ_r. But this ratio is also the ratio of the velocities of light in the respective media and is therefore inversely proportional to the indices of refraction.

Therefore,

$$n_i \sin\theta_i = n_r \sin\theta_r$$

as was to be proved.

There are other ways to prove Snell's law, but the visual appeal of a geometric proof is that the involved quantities and their relationships can be easily seen in the figure.

20 Mirrors and Lenses

This section contains proofs for a few important theorems in optics.

20.1 Finding the Focal Point

The *focal point* of a concave mirror is that point at which light rays from a distant object are expected to converge. It is a central concept in the characterization of both mirrors and thin lenses.

For our derivation, we observe that it is only necessary to consider a cross-section of the mirror which passes through the center of curvature. By symmetry, the behavior of distant rays which strike the mirror elsewhere will be the same.

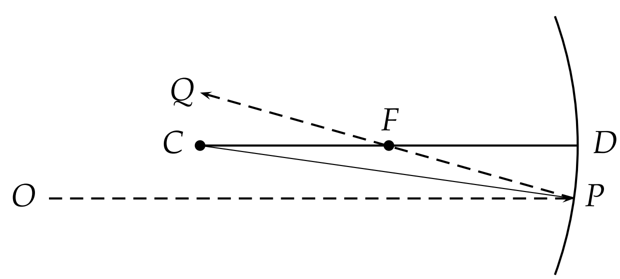

Figure 20.1: Ray Diagram

In the diagram, C is the center of the sphere which forms the contour of the mirror. $\overline{CD}$ is a radius to the center of the arc. A ray from a distant object, $\overline{OP}$, parallel to $\overline{CD}$ is reflected as ray $\overline{PQ}$ intersecting $\overline{CD}$ at F.

$\overline{CP}$ is a radial line from the center of the sphere and is therefore normal to the surface at P. The angle of incidence is equal to the angle of reflection, so $\angle OPC = \angle CPQ$. Let the angle of incidence = α. Then $\angle CPQ = \alpha$ and $\angle PCD = \alpha$. $\angle PFD = 2\alpha$.

Now for the approximations. Assume the ray $\overline{OP}$ is very close to $\overline{CD}$. In this case the arc $\overline{DP}$ is short and very nearly a straight line perpendicular to $\overline{CD}$. With this approximation, $\tan(\angle DFP) = \tan(2\alpha) \approx \frac{\overline{DP}}{\overline{FD}}$. Also, $\tan(\angle DCP) = \tan(\alpha) \approx \frac{\overline{DP}}{\overline{CD}}$.

With $\overline{OP}$ close to $\overline{CD}$ the angles α and 2α are small. Hence, $\tan(\alpha) \approx \alpha$ and $\tan(2\alpha) \approx 2\alpha$. So $\frac{\overline{DP}}{\overline{FD}} \approx 2\frac{\overline{DP}}{\overline{CD}}$, or $\overline{CD} \approx 2\overline{FD}$. This places point F at the midpoint of radius $\overline{CD}$.

We have now shown that rays from distant objects whose paths are parallel to and sufficiently close to the radius through the center of a spherical mirror will, after reflection, pass through a common point, F, whose distance from the mirror is 1/2 the radius. This point is called the *focal point*.

In the literature, the focal point is usually identified with the symbol f, and the equation

$$f = \frac{r}{2},$$

where r is the radius of curvature is used.

20.2 The Mirror/Lens Equation

A central equation in optics relates the *focal length*, *object distance* and *image distance* for a thin lens or mirror. This equation is usually expressed as

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}, \tag{20.1}$$

where f is the focal length, p is the distance from the object to the lens, and q is the distance from the lens to the image.

We can derive this equation from the following figure using simple geometry.

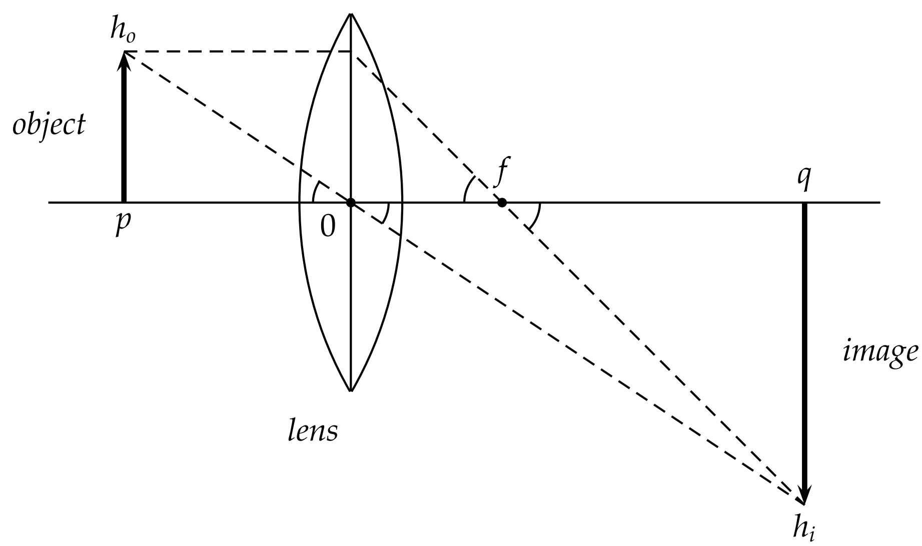

Figure 20.2: Lens, Object and Image

In the figure, the center of the lens is at 0. p is the distance from the lens to the object and q is the distance from the lens to the image. h_o is the height of the object and h_i is the height of the image. f is the distance from the lens to the focal point.

Using the similar triangles indicated by the alternate interior angles at 0, consisting of sides p and h_o for the left and q and h_i for the right, we find

$$\frac{h_o}{h_i} = \frac{p}{q}. \tag{20.2}$$

Likewise, from the similar triangles indicated by the alternate interior angles at f, consisting of h_o along the center line of the lens and f for the left and h_i and $q - f$ for the right, we find

$$\frac{h_o}{h_i} = \frac{f}{q - f}. \tag{20.3}$$

Hence,

$$\begin{aligned}
\frac{p}{q} &= \frac{f}{q-f} && (20.4)\\
p\cdot q - p\cdot f &= q\cdot f && (20.5)\\
p\cdot q &= f(p+q) && (20.6)\\
f &= \frac{p\cdot q}{p+q} && (20.7)
\end{aligned}$$

Finally,

$$\frac{1}{f} = \frac{p+q}{p\cdot q} = \frac{1}{p} + \frac{1}{q}. \qquad (20.8)$$

.

20.3 Lensmaker's Formula

Lenses with the same shape and index of refraction will have the same focal length. the *lensmaker's formula* relates the index of refraction, the radii of curvature of the two surfaces of the lens, and the focal length of the lens.

A number of idealizations, simplifications and approximations are used to complete the derivation, but the results are compact and sufficiently accurate for most purposes.

We begin by observing that a lens with convex surfaces behaves the same as two plano-convex lenses placed with the flat sides in contact. Fig. (20.3) shows the division of the lens into two pieces which we will analyze separately.

Recall that with thin lenses we can reverse the direction of the ray without affecting the incident and refracted angles. Hence, Fig. (20.4) which represents one plano-convex lens may be regarded as the rightmost half of the original lens or the leftmost half reversed. In this figure, a perpendicular ray enters the flat surface of the lens. It proceeds to the curved surface without initial refraction. When it emerges from the curved surface it is refracted by an angle determined by Snell's law. The radius from the center of curvature extended through the exit point determines the surface

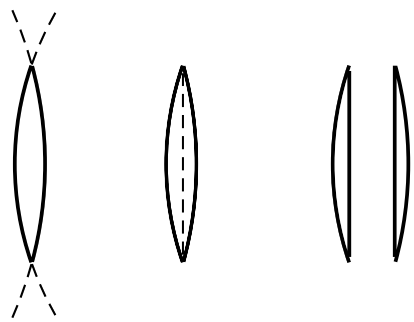

Figure 20.3: Separation of Lens into Halves

normal. The angle in the media between the ray and the normal is θ_1. The angle between the refracted ray and the normal is θ_2.

If the index of refraction of the lens is n and we take the index of refraction of air as 1, Snell's law holds that

$$n \sin\theta_1 = \sin\theta_2.$$

Assuming small angles (paraxial rays), we now approximate the sines of the angles with the angles themselves so that

$$n\,\theta_1 \approx \theta_2.$$

Substituting this in the angle between the refracted ray and the axis

$$\theta_2 - \theta_1 = n\,\theta_1 - \theta_1 = (n-1)\theta_1. \tag{20.9}$$

For these small angles, the tangents are also close to the angles themselves. We can write

$$\theta_2 - \theta_1 \approx \frac{h}{f_1}, \tag{20.10}$$

and

$$\theta_1 \approx \frac{h}{R_1}. \tag{20.11}$$

Eliminating h between (20.10) and (20.11) and substituting from (20.9),

$$\frac{1}{f_1} = \frac{n-1}{R_1}. \tag{20.12}$$

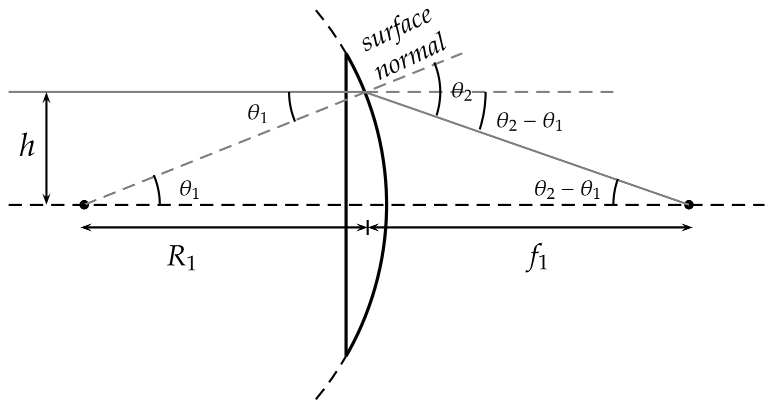

Figure 20.4: Ray Diagram for Lens Analysis

Substituting from the lens equation (20.8) which relates the object and image distances to the focal length

$$\frac{1}{o_1} + \frac{1}{i_1} = \frac{n-1}{R_1}. \tag{20.13}$$

An equivalent analysis of the other half of the lens gives

$$\frac{1}{o_2} + \frac{1}{i_2} = \frac{n-1}{R_2}. \tag{20.14}$$

We can now combine (20.13) and (20.14) noting that the image of the first lens is a virtual object for the second lens. Therefore $i_1 = -o_2$ and, adding the two equations,

$$\frac{1}{o_1} + \frac{1}{i_2} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{20.15}$$

Writing the lens equation in terms of the object and image distances,

$$\frac{1}{o} + \frac{1}{i} = \frac{1}{f}. \tag{20.16}$$

But o_1 and i_2 are the object and image distances of the whole lens, so $o_1 = o$ and $i_2 = i$. Thus,

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{20.17}$$

which is the **lensmaker's formula**.

Considering the approximations used, we should not expect this formula to be accurate for large angles of incidence, but for many purposes it is quite useful.

21 Solar Constant

The luminosity of the Sun, L, is 3.827×10^{26} Watts. When the Sun is directly overhead on a clear day, how many watts would we expect to illuminate a square meter of the Earth's surface?

This value is found by spreading the luminous flux over an imaginary sphere with its center at the center of the Sun and its radius equal to the Earth's distance from the Sun, 1.496×10^{11} meters.

The sphere has a surface area equal to

$$A = 4\pi r^2 = 4\pi(1.496 \times 10^{11})^2 \qquad (21.1)$$

$$A = 2.81 \times 10^{23}\ m^2. \qquad (21.2)$$

Then the value we seek is L/A or 1361 W/m^2. This is called the Solar Constant and is most useful in analyzing solar powered equipment and evaluating energy conservation methods.

22 Miscellaneous Physical Constants

The use of parentheses indicate that no standard symbol has been established. Empty parentheses indicate no suggested symbol.

Constant	Symbol	Value
Speed of light in vacuum	c	2.9979×10^{8} m/sec
Gravitational constant	G	6.673×10^{-11} nt-m^2/kg^2
Planck constant	h	6.6262×10^{-34} joule-sec
Electron charge	e	1.6021×10^{-19} coulomb
Electron rest mass	(m_e)	9.1086×10^{-31} kg
Proton rest mass	(m_p)	1.6724×10^{-27} kg
Boltzmann's constant	k	1.308×10^{-23} J/K
Radius of earth	(R_e)	6370 km
Mass of earth	(M_e)	5.98×10^{24}kg
Distance from Earth to Sun	()	1.496×10^{8} km
Diameter of Sun	()	6.95×10^{5} km
Mass of Sun	()	1.989×10^{30} kg
Luminosity of Sun	()	3.827×10^{26} W
Solar constant	()	1358 W/m^2
Distance from Earth to Moon	()	3.844×10^{5} km
Radius of Moon	()	1738 km
Mass of Moon	()	7.348×10^{22} kg