Notes on the 'Q' Function of Williams and Comstock

by C. Bond

1 Background

Because of some poorly documented differences between the descriptions of the 'Q' function in the original Williams & Comstock paper[1] and the book by Neal Bertram[2] I decided to re-derive the equations on which the W&C model is based. The intent was to explain and resolve those differences. On completing the analysis, I found that Bertram describes a more general formulation of the Q function which plots a family of curves, rather than the single plot of W&C, which assumes an optimal head field value. This difference clearly impacts the design of head/media geometries for specific write applications, as will be shown later in this paper.

In the following paragraphs, I follow the original modeling strategy presenting explicit details of the derivation of the equations and the resulting relations. In addition, I present an exact closed form expression for the W&C version of the Q function — which to my knowledge has not been previously published. Limiting values for Q at $x = 0$ and $x = \infty$ are given. Further, I will show that the W&C model predicts adequate write capability for permalloy at media coercivities approaching 3000 Oe — which raises some questions about either the model or the conventional wisdom.

2 Summary

According to Williams & Comstock, an optimal recording transition (minimal length) will be written if the deep gap field of the write head is adjusted so that the maximum gradient exists where the external head field equals the coercivity of the media at the given head/media spacing. Their modeling strategy, which assumes a Karlquist head field, requires finding the maximum field gradient for a given geometry as a function of the 'x' (downtrack) distance from the gap. This gradient is a function of the gap length, g, coercivity, H_c, and spacing (fly height), y, only.

$Q(y/g)$ is defined as a function of y/g and is associated with the maximum write field gradient such that the equation:

$$\frac{dHh}{dx} = \frac{-Q\,H_c}{y} \tag{1}$$

is optimal under the stated conditions. Q is neither a function of H_c nor H_0 as will be shown.

3 Karlquist Head Field

The Karlquist head field is given by:

$$H_h = H_k = \frac{H_0}{\pi}\left(\arctan\left(\frac{x+g/2}{y}\right) - \arctan\left(\frac{x-g/2}{y}\right)\right), \tag{2}$$

where g is the gap length, y is the head to media spacing, and H_0 is the deep gap field. This equation is of the form $H_h = H_0(\theta_1 - \theta_2)/\pi$, where θ_1 and θ_2 are the angles between some field point external to the head and the two gap edges.

[1] Williams & Comstock, *An Analytical Model of the Write Process in Digital Magnetic Recording*, 17th Annu. Conf. Proc. Part 1, No. 5, 1971, pp. 738-742

[2] Bertram, H. Neal, *Theory of Magnetic Recording*, 1994, Cambridge University Press

Note that we are placing the vertical axis along the centerline between the inside gap edges. W&C set the axis along the trailing edge. This only affects the algebra, not the result.

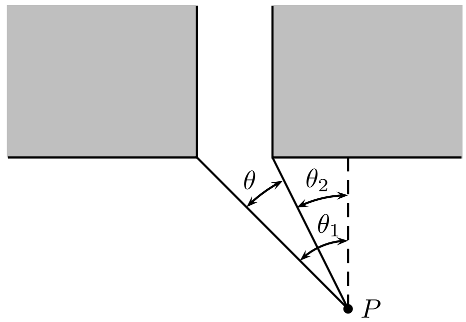

Figure 1: Karlquist Geometry

The geometry is shown in Fig. (1).

Using the trigonometric identity

$$\arctan(A) - \arctan(B) = \arctan(\frac{A-B}{1+AB}) \tag{3}$$

Eq. (2) can be simplified to

$$H_h = \frac{H_0}{\pi} \arctan\left(\frac{g\,y}{x^2+y^2-(g/2)^2}\right) = \frac{H_0}{\pi}\theta, \tag{4}$$

where θ is the angle *subtended* by the field point and the two gap edges.

W&C address the write process, which is constrained to take place at the point in the media where the head field equals the media coercivity. They begin by solving the Karlquist equation to find the value of x for which $H_h = H_c$. They require that the head field be adjusted so that the maximum gradient is there, as common sense would suggest.

From Eq. (4) we take $\theta = \pi\, H_h/H_0$ so,

$$\frac{g\,y}{x_o^2+y^2-(g/2)^2} = \tan\left(\frac{\pi\, H_c}{H_0}\right), \tag{5}$$

where x_0 is the value of x for which $H_h = H_c$.

The last equation can be solved for x_0 yielding

$$x_0 = \pm\frac{g}{2}\sqrt{1-\left(\frac{2y}{g}\right)^2 + \frac{4y}{g\tan\left(\pi H_c/H_0\right)}}, \tag{6}$$

where the two solutions correspond to the two points in the media where the conditions are satisfied. We will use the positive solution in the following analysis.

4 The Gradient

The Karlquist equation is a function of several variables, and we must be mindful when taking derivatives. Some of the 'fixed' variables may be subject to later changes.[3]

[3]It is a fairly common practice to treat partial differential quantities as total differentials. Of course, this will work if all other variables are assigned constant values for a given experiment, but the practice *is* questionable and can lead to errors. (A mathematician and a physicist came upon a house on fire...)

For the purposes at hand, the spacial gradient of the head field is

$$\frac{dH_h}{dx} = \frac{H_0}{\pi}\frac{d\theta}{dx}. \tag{7}$$

Differentiating θ

$$\frac{dH_h}{dx} = \frac{H_0}{\pi}\left(\frac{1}{1+\left(\frac{yg}{x^2+y^2-(g/2)^2}\right)^2}\right)\frac{-2xyg}{(x^2+y^2-(g/2)^2)^2}, \tag{8}$$

which simplifies to

$$\frac{dH_h}{dx} = \frac{-2x\,H_0\,yg}{\pi((x^2+y^2-(g/2^2))^2+(yg)^2)}, \tag{9}$$

or even better, using the identity

$$\frac{\tan^2(\omega)}{1+\tan^2(\omega)} = \sin^2(\omega), \tag{10}$$

we find

$$\frac{dH_h}{dx} = \frac{-2x\,H_0}{\pi\,yg}\sin^2(\theta). \tag{11}$$

5 Maximizing the Gradient

To find the value of x which corresponds to the maximum gradient, set the equation for the derivative of the gradient to zero and solve for x. The resulting equation will *not* be a function of H_0 because H_0 appears in the starting Karlquist equation as a (constant) multiplier only, and all multipliers appear in the derivatives as multipliers. Hence, setting the second derivative of the Karlquist equation to zero effectively removes H_0 (and π) from the result.

We find that the value of x for maximum gradient is given by

$$x_m = \frac{\sqrt{g^2-4y^2+2\sqrt{g^4+4g^2y^2+16y^4}}}{2\sqrt{3}}, \tag{12}$$

and is a function of y and g only.

We can solve Eq. (6) for H_0 by substituting the value for x_m from Eq. (12) for x_0. In doing this, we satisfy the condition that the maximum gradient occurs at the point of the transition in the media. The resulting value for H_0 is

$$H_{opt} = \frac{H_c\pi}{\arctan\left(\frac{6gy}{4y^2-g^2+\sqrt{g^4+4g^2y^2+16y^4}}\right)}. \tag{13}$$

We can now find an expression for the maximum gradient in terms of H_c, g and y by substituting x_m for x and H_{opt} for H_0 in Eq. (8). Thus

$$\left.\frac{d\,H_h}{dx}\right|_{\max} = \frac{6\sqrt{3}\,g\,H_c y\sqrt{g^2-4y^2+2p}}{(p(g^2-4y^2)-g^4-16g^2y^2-16y^4)\arctan\left(\frac{6gy}{p-g^2+4y^2}\right)}, \tag{14}$$

where $p = \sqrt{g^4+4g^2y^2+16y^4} = g^2\sqrt{1+4y^2/g^2+16y^4/g^4}$.

Note that the role of th Q function in the W&C paper is to facilitate the calculation of the maximum field gradient under specified conditions. From Eq. (14) we have derived and expression for the maximum gradient already, and the need for an auxiliary function (the Q function) is less compelling. Nevertheless, to complete the analysis, we now derive a formula for Q.

6 The Williams & Comstock Q Function

The Q function is obtained by factoring out that part of the expression for the maximum gradient which is a function of y/g. Referring to Eq. (8) we have

$$Q = -\frac{y}{H_c}\left.\frac{d\,H_h}{dx}\right|_{\max} \tag{15}$$

The expression obtained by expanding the right side of Eq. (15) directly can be rearranged to show more explicity the dependence of y/g giving

$$Q = \frac{-6\sqrt{3}(y/g)^2\sqrt{1+2m-4(y/g)^2}}{(((1-4(y/g)^2)m-1-16(y/g)^2+16(y/g)^4)\arctan\left(\frac{6(y/g)}{m+4(y/g)^2-1}\right)}, \tag{16}$$

where $m = \sqrt{1+4(y/g)^2+16(y/g)^4}$.

The closed form expression for Q in Eq. (16) shows that it is, as described in the original paper, a functions of y/g (only). Further, a plot Q is given below over the same range is that in the W&C paper, confirming that this is the same function they described. It can be shown that Q has a limiting value of $2/\pi$ at $y/g = 0$

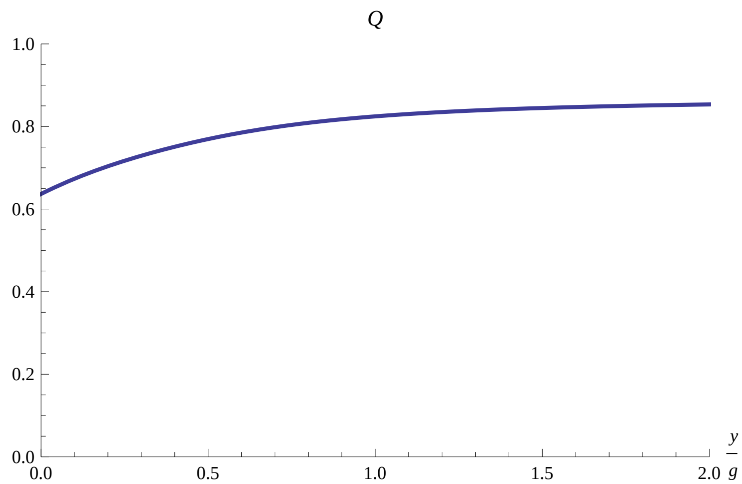

Figure 2: Williams-Comstock Q Function vs. y/g

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and $\sqrt{3(\sqrt{15}-2)/2}/2$ as $y/g \to \infty$. There are no finite maxima. Hence $0.6366 < Q < 0.8381$ for all positive y and g. It is very easy to devise simple approximations which are more convenient for casual use.

Using the expression already found for x_0, we can also express x_0/g as a function of y/g only. In fact,

$$\frac{x_0}{g} = \frac{\sqrt{1-4(y/g)^2+2\sqrt{1+4(y/g)^2+16(y/g)^4}}}{2\sqrt{3}} \tag{17}$$

A plot of x_0 and Q is shown in Fig. (3).[4]

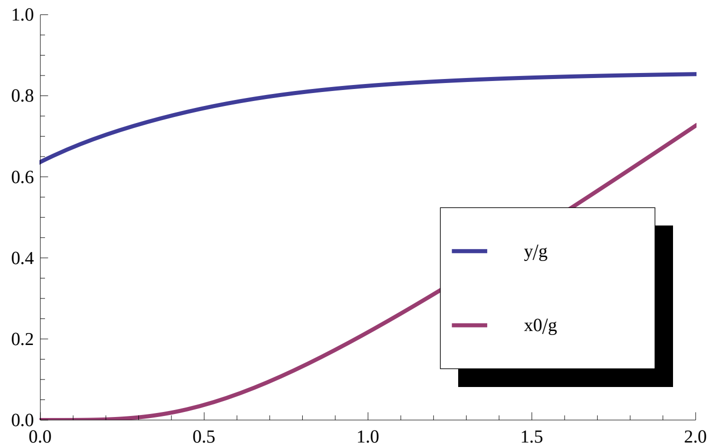

Figure 3: Q Function and x_0/g vs. y/g

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7 Bertram's Q Function

To reduce potential confusion, I will refer to the Bertram version of the Q function as Q_B. It will be shown that this is a different formulation yielding different results from those of W&C.

The Q function previously derived is single valued for every y, g and assumes that the point of maximum head field gradient has set into the media. Bertram, on the other hand, decouples the Q which appears in Eq. (1) and treats it as a general function of y, g, H_0 and H_c. In his treatment, Q is not constrained to a fixed relation between H_0 and H_c.

The starting point for understanding the different results provided by the two approaches is

$$\left.\frac{d\,H_h}{dx}\right|_{\max} = -\frac{Q\,H_c}{y} \qquad \text{W\&C} \tag{18}$$

$$\frac{d\,H_h}{dx} = -\frac{Q_B\,H_c}{y} \qquad \text{Bertram.} \tag{19}$$

The explicit form given by Bertram for Q is

$$Q_B = \frac{2x_0 H_0}{\pi g H_c}\sin^2(\pi H_c/H_0), \tag{20}$$

which is a function of y, g, H_c and H_0, rather than a function of y and g only. The reason for uncoupling H_c and H_0 is that the gradient in the media may be increased by increasing the field. This moves the maximum head field gradient out of the media, but may increase the gradient in the media nevertheless.

Because Bertram redifines Q, he must adopt a different optimization strategy. Instead of tracking the point of maximum head field gradient and adjusting the field until this point is in the media, he adjusts the head field for maximum gradient in the media. Note that once the peak head field exceeds H_c further increases of H_0 will move the switching point. $H_x = H_c$ further from the gap.

Plots for Q_B with various y/g ratios are shown in Fig. (4).

[4] This is one instance where the different coordinate system of Williams & Comstock *w.r.t.* the location of the x-axis origin makes a difference. In my plot, I had to lower the plot of x_0 by $g/2$ in order to match the plot given in the W&C paper.

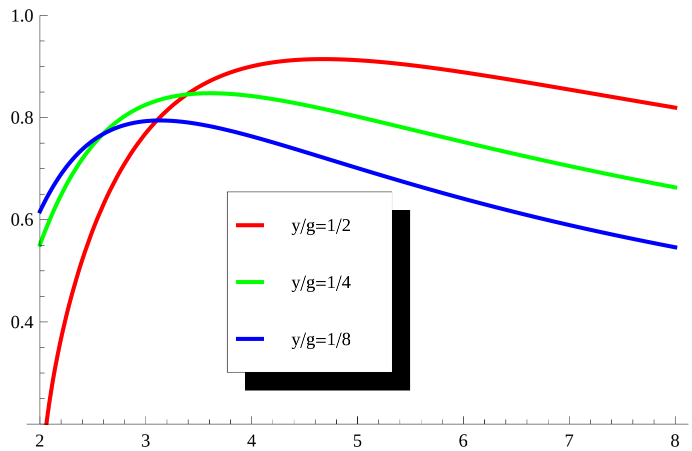

Figure 4: Bertram's Q Function

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As previously stated, this strategy yields different results from those of W&C. The following table shows a comparison of the results for a specific geometry. Note that the field value, H_0, for Bertram is above the comfort level for permalloy, but that the W&C result is comfortable. Bertram's strategy for maximizing the gradient extracts a penalty — improving the gradient by 18% over the W&C value requires an increase in the field by 51%! Any application of Bertram's model clearly requires careful analysis.

Model	g	y	H_c	x_0	$Q\,(Q_B)$	H_0	H_0/H_c	$d\,H_0/dx$
W&C	0.2	0.05	2400	0.101	0.717	5742	2.4	-34416.5
Bertram	0.2	0.05	2400	0.126	0.848	8660	3.6	-46689.2

8 Conclusion

Bertram describes a different method for optimizing the head field gradient from that of W&C. The intent in both cases is to minimize the transition length or a parameter. But the differences are not trivial and would lead to different design decisions concerning optimized head/media combinations.

The W&C strategy involves finding the minimum field required to support a particular gradient in the media, although this is not directly stated in their paper. Once a gradient has been calculated by their method, no lower field can produce the same gradient.

Bertram's method, on the other hand, involves finding the maximum possible gradient for $H_h = H_c$, with no constraint on the field required to obtain it.

Because of the larger fields required to satisfy Bertram's method, there may be a significant exposure to non-linear effects in the media. NLTS and partial erase problems could be exacerbated at high densities, and there may be a need for higher B_{sat} materials.

These observations, results and conclusions are solely for the purpose of comparing to two optimization strategies. No three dimensional or high frequency effects have been considered.